TMP-TC2: Cosmology

Problem Set 5 23, 24, 25 May 2023

1. Basics of Thermodynamics

The number density n, the energy density ρ and the pressure p are respectively given by

$$n = \frac{g}{(2\pi)^3} \int f(\vec{k}) \mathrm{d}^3 k \ , \tag{1}$$

$$\rho = \frac{g}{(2\pi)^3} \int E(\vec{k}) f(\vec{k}) \mathrm{d}^3 k \ , \tag{2}$$

$$p = \frac{g}{(2\pi)^3} \int \frac{|\vec{k}^2|}{3E} f(\vec{k}) \mathrm{d}^3 k \ , \tag{3}$$

with g the number of degrees of freedom and $f(\vec{k})$ the distribution function which is given by

$$f(\vec{k}) = \frac{1}{e^{\frac{E-\mu}{T}} \pm 1} \ , \tag{4}$$

where $+$ is for fermions and $-$ for bosons.

1. Derive the expression for the pressure p for an arbitrary distribution function $f(\vec{k})$.
2. Take the limit $T \gg |\mu|$ and $T \gg m$ and show that for fermions

$$n = \frac{3}{4\pi^2} \zeta(3) g T^3 \ , \quad \rho = \frac{7}{8} \frac{\pi^2}{30} g T^4 \ . \tag{5}$$

3. In the same limit, show that for bosons

$$n = \frac{1}{\pi^2} \zeta(3) g T^3 \ , \quad \rho = \frac{\pi^2}{30} g T^4 \ . \tag{6}$$

4. Take this limit to show that

$$p = \frac{1}{3} \rho \tag{7}$$

5. **(Bonus)** In the limit $T \gg m$ but $\mu \neq 0$ the difference between the number of particles and anti-particles is non-zero. What is $n - \bar{n}$ and $\rho + \bar{\rho}$ for fermions? You can assume that the chemical potential of an anti-particle is equal to minus the chemical potential of the corresponding particle; we shall prove this statement on the next sheet.
   Hint : You may use Mathematica to solve the integrals.

6. Show that in the non-relativistic limit the number density boils down to

$$n = g \left(\frac{mT}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{m-\mu}{T}} \ , \tag{8}$$

both for fermions and bosons.

2. Effective number of degrees of freedom

Consider the phase of the Universe when electroweak phase transition had not yet taken place, i.e. for $T > 200GeV$. In that case we can assume that $m \approx 0$ and $|\mu| \approx 0$ for all particles. Therefore, the energy density is given by

$$\rho(T) = \rho_{\text{fermions}}(T) + \rho_{\text{bosons}}(T) = \frac{\pi^2}{30} g_* T^4 \ . \tag{9}$$

Find the value of g_* assuming that all the Standard Model particles contribute.

3. The Entropy of the Universe

1. Assuming $|\mu_i| \ll T$ and $m \ll T$, show that

$$s \equiv \frac{S}{V} = \frac{2\pi^2}{45} g_* T^3 \ , \tag{10}$$

   where g_* is the effective number of degrees of freedom and S is the entropy. Use the Euler equation $E = TS - pV + \sum_i \mu_i N_i$.
2. Assuming that the main contribution to the current entropy density is given by relic photons, estimate the total entropy of the visible part of the Universe. *Indication :* The size of the visible part of the Universe is $l_0 \sim 10^{28} cm$ and the todays Cosmic Microwave Background (CMB) temperature is $T_0 \approx 2.73K$.
3. Using the first principles of thermodynamics and the continuity equation, show that the entropy of the universe is conserved if we assume that the Universe is homogeneous and $\mu = 0$. Remember that for a homogeneous system the Euler equation holds.

4. Relation between time and temperature

In this problem, we shall establish a valuable connection that correlates the age of the universe with its corresponding temperature. We shall work in the limit $T \gg |\mu|, m$.

1. Assuming that the entropy of the Universe is conserved, show that $T \propto g_*^{-\frac{1}{3}} R^{-1}$.
2. Using the first Friedmann equation for a flat Universe, derive the relation

$$T \approx 0.55 g_*^{-\frac{1}{4}} \sqrt{\frac{M_P}{t}} \approx 1.56 \ g_*^{-\frac{1}{4}} \sqrt{\frac{1\text{s}}{t}} \ \text{MeV} \ . \tag{11}$$