TMP-TC2: Cosmology

Solutions to Problem Set 4 16, 17, 18 May 2023

1. Comoving Distance and Redshift

1. From Problem Set 2 we know that

$$\begin{aligned} \chi &= \operatorname{arsinh} r && \text{for } k=-1 \\ \chi &= r && \text{for } k=0 \\ \chi &= \arcsin r && \text{for } k=+1 \end{aligned}$$

2. Photons follow null geodesics and thus

$$\mathrm{d}s^2 = 0 = \mathrm{d}t^2 - R(t)^2 \mathrm{d}\chi^2 \tag{1}$$

From here we obtain

$$\chi = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{R(t)} \tag{2}$$

3. The redshift is defined as

$$1 + z = \frac{\lambda_2}{\lambda_1} \ , \tag{3}$$

with λ_1 and λ_2 the wavelengths at the time of emission and observation, respectively. The wavelengths increase due to the expansion of the Universe

$$\lambda \propto R \ , \tag{4}$$

meaning that

$$1 + z = \frac{R(t_2)}{R(t_1)} \ .$$

4. We start from the definition of the Hubble parameter

$$H = \frac{\dot{R}}{R} = \frac{1}{R}\frac{dR}{dt} = \frac{1}{R}\frac{dR}{dz}\frac{dz}{dt} \ .$$

Using $1 + z = \frac{R_0}{R}$, the above becomes

$$H = -\frac{1}{dt}\frac{dz}{(1+z)} \ .$$

Therefore, for the age of the universe we find

$$t = \int_0^t \mathrm{d}t = \int_z^\infty \frac{\mathrm{d}z'}{(1+z')H} \tag{5}$$

5. For a matter dominated universe we have $R \propto t^{\frac{2}{3}}$. Therefore, the Hubble constant $H \propto R^{-\frac{3}{2}}$ and we obtain

$$\frac{H}{H_0} = \left(\frac{R_0}{R}\right)^{\frac{3}{2}} = (1+z)^{\frac{3}{2}} \tag{6}$$

The age of the universe is given by $t(z=0)$. Hence, the time Δt that was passed since emission is

$$\begin{aligned}\Delta t &= t(z=0) - t(z) \\ &= \int_0^z \frac{\mathrm{d}z'}{(1+z')^{\frac{5}{2}} H_0} \\ &= \frac{1}{H_0}\left(-\frac{2}{3}\frac{1}{(1+z)^{\frac{3}{2}}} + \frac{2}{3}\right)\end{aligned}$$

Inserting the numbers gives $\Delta t \approx 2.28 \cdot 10^{17} s \approx 7.23$ billion years. The distance at the time of emission is

$$\begin{aligned}d &= R(t_{em})\chi \\ &= 3t_{em}^{\frac{2}{3}}(t_0^{\frac{1}{3}} - t_{em}^{\frac{1}{3}}) \\ &= 3(t_0 - \Delta t)^{\frac{2}{3}}(t_0^{\frac{1}{3}} - (t_0 - \Delta t)^{\frac{1}{3}})\end{aligned}$$

With $t_0 = \frac{2}{3H_0}$ we obtain

$$\begin{aligned}d &\approx 1.19 \cdot 10^{17} s \cdot c \\ &\approx 3.57 \cdot 10^{22} \text{ km} \\ &\approx 3.77 \text{ billion light years}\end{aligned}$$

2. Evolution of the Universe

1. Recall that

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}, \quad \rho_\Lambda = \frac{\lambda}{8\pi G}, \quad \rho_c = \frac{3H_0^2}{8\pi G}. \tag{7}$$

We also know that $1Mpc = 3.1 \cdot 10^{24} cm$. In natural units,

$$1cm = 5 \cdot 10^{13} GeV^{-1}, \quad G = M_P^{-2}, \quad M_P = 1.2 \cdot 10^{19} GeV. \tag{8}$$

Assuming $\Omega_\Lambda = 0.7$ and $H_0 = 73\frac{km}{s \cdot Mpc}$, we have

$$\rho_c \approx 0.5 \cdot 10^{-5}\frac{GeV}{cm^3}. \tag{9}$$

Hence

$$\rho_\Lambda \approx 0.35 \cdot 10^{-5}\frac{GeV}{cm^3}, \tag{10}$$

and

$$\lambda = \frac{8\pi\rho_\Lambda}{M_P^2} \approx 5 \cdot 10^{-84} GeV^2. \tag{11}$$

2. The present radiation density is :

$$\rho = \frac{\pi^2}{30} g T^4 = \frac{\pi^2}{30} g \frac{T^4 k_B^4}{\hbar^3 c^5} = 4.7 \cdot 10^{-31} [kg/m^3].$$

The critical density ($H = 70 \frac{km}{s\ Mpc}$) is

$$\rho_c = \frac{3H^2}{8\pi G} = 9.2 \cdot 10^{-27} [kg/m^3],$$

and then

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c} = 5 \cdot 10^{-5}.$$

It is a flat space, $\Omega_m + \Omega_\lambda = 1$, $\Omega_\gamma \sim 0$.

3. We solve the Friedmann equations now, returning in time (t_0 is the current age of the universe). According to the approximation proposed, we take into account only λ, and the evolution of the universe is given by

$$R(t) = R_0 e^{\sqrt{\frac{\lambda}{3}}(t - t_0)},$$

until the matter domination. Indeed, if the universe becomes smaller, matter concentrates ($\rho_m \propto R^{-3}$) while the cosmological constant does not change [1]. The point $(t_\lambda, R_\lambda = R(t_\lambda))$ where the densities are equal

$$\Omega_\lambda(t_0) = \Omega_\lambda(t_\lambda) = \Omega_m(t_\lambda) = \Omega_m(t_0) \frac{R_0^3}{R_\lambda^3},$$

is for

$$\frac{R_\lambda}{R_0} = \left(\frac{\Omega_m(t_0)}{\Omega_\lambda(t_0)} \right)^{1/3} = e^{\sqrt{\frac{\lambda}{3}}(t_\lambda - t_0)}.$$

Then

$$t_\lambda - t_0 = (3\lambda)^{-1/2} \ln \frac{\Omega_m}{\Omega_\lambda}.$$

By using $\lambda = 3\Omega_\lambda H^2 = 1.1 \cdot 10^{-35}$, we find $t_\lambda - t_0 = -4.58 \cdot 10^9$ years. We check at this time that $\rho_\gamma \ll \rho_m$, in fact, $\rho_\gamma(t_\lambda) = \rho_\gamma(t_0)(R_0/R_\lambda)^4 = 3.1 \rho_\gamma(t_0)$. The evolution of the universe dominated by the mater is

$$R(t) = R_0 \left(\frac{t}{t_0} \right)^{2/3}.$$

Since $\rho_\gamma \sim R^{-4}$ and $\rho_m \propto R^{-3}$ the radiation had dominated for t_m, R_m very small. The transition is when

$$\Omega_m(t_m) \quad = \quad \Omega_\gamma(t_m)$$

1. The energy density ρ_Λ of the cosmological constant is constant. In the phase where the universe is dominated by the cosmological constant the critical density is also constant, and also the abundance Ω_Λ.

As we just mentioned

$$\Omega_m(t_m) = \Omega_m(t_\lambda)\frac{R_\lambda^3}{R_m^3} \qquad \text{et} \qquad \Omega_\gamma(t_m) = \Omega_\gamma(t_\lambda)\frac{R_\lambda^4}{R_m^4}.$$

Since $R \sim t^{2/3}$, we get

$$\frac{\Omega_m(t_\lambda)}{\Omega_\gamma(t_\lambda)} = \frac{R_\lambda}{R_m} = \frac{t_\lambda^{2/3}}{t_m^{2/3}}.$$

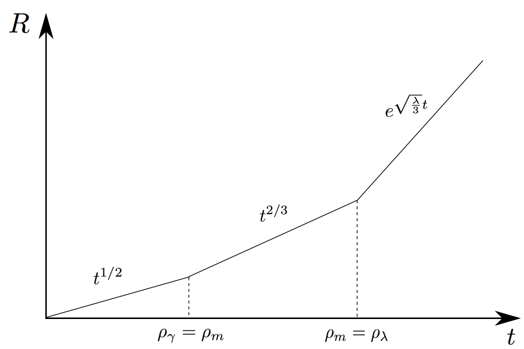
We can write the L.H.S. as

$$\frac{\Omega_m(t_\lambda)}{\Omega_\gamma(t_\lambda)} = \frac{\Omega_m(t_0)}{\Omega_\gamma(t_0)}\frac{R_\lambda}{R_0} = \frac{\Omega_m(t_0)}{\Omega_\gamma(t_0)}\left(\frac{\Omega_m(t_0)}{\Omega_\lambda(t_0)}\right)^{1/3},$$

then $t_m = 3.3 \cdot 10^{-6} t_\lambda$. Earlier, the universe was dominated by radiation

$$R(t) = R_m(t/t_m)^{1/2}.$$

In conclusion, we have $t_\lambda = 9.52 \cdot 10^9\ y$, $t_m = 31'000\ y$. The present age of the universe is $t_0 \simeq 14 \cdot 10^9 y$. The matter had dominated until "recently" and almost from the beginning.



3. Dipole anisotropy of the Cosmic Microwave Background

We want to show how the motion of the Earth inside the CMB gives rise to anisotropy in the temperature spectrum. To this end, it is useful to work with Lorentz invariant quantities. A Lorentz scalar is for example the total number of photons, given by

$$N = \int \mathrm{d}^3p\ \mathrm{d}^3x\ f(\omega, T) \tag{12}$$

where $f(\omega, T)$ is the photon number density in phase-space, given by

$$f(\omega, T) = \frac{1}{e^{\frac{\omega}{kT}} - 1} \tag{13}$$

Since N and $\mathrm{d}^3p\mathrm{d}^3x$ are Lorentz scalars, the number density $f(\omega, T)$ must also be a Lorentz scalar.
If we move to a coordinate system that is moving with respect to the CMB

$$\omega_{CMB} = \frac{1 + v\cos\theta}{\sqrt{1 - v^2}}\omega_{OBS}\ , \tag{14}$$

where θ is the angle betweenthe direction of observation and Earth's velocity. For $f(\omega, T)$ to be invariant, the temperature must transform as

$$T_{OBS} = \frac{\sqrt{1-v^2}}{1+v\cos\theta} T_{CMB} \ . \tag{15}$$

Expanding the above in powers of v and keeping the first order term, we find

$$T_{OBS} \approx (1 - v\cos\theta) T_{CMB} \quad \rightarrow \quad \frac{\delta T}{T} \approx -v\cos\theta \ . \tag{16}$$

Considering a dipole anisotropy of the order of 10^{-3}, we find for $\theta = \pi$

$$v \approx 10^{-3} \ , \tag{17}$$

or, in conventional units,

$$v \approx 370 \text{ km/s} \ . \tag{18}$$

4. Photon decoupling in numbers

1. The redshift is defined as

$$1 + z(t) = \frac{R_0}{R(t)} \ , \tag{19}$$

where $R_0 = R(t_0)$ is the scale factor today. Since temperature redshifts as $T \sim R^{-1}$, we find that at the time of decoupling

$$z(t_d) = \frac{T_d}{T_0} - 1 \approx 1100 \ . \tag{20}$$

2. In order to compute the age of the Universe

$$t = \int dt \ ,$$

when photons decoupled, we have to express the above in terms of the redshift. Time and redshift are related as

$$t = \int \frac{dz}{(1+z)H} \ .$$

We have seen that

$$H^2 = H_0^2 \left[\Omega_\lambda + (1+z)^3 \Omega_m + (1+z)^4 \Omega_\gamma \right] \ .$$

Using this, the time of decoupling is obtained as

$$t_d = \frac{1}{H_0} \int_0^{z_d} \frac{dz'}{(1+z')\sqrt{\Omega_\lambda + \Omega_m (1+z')^3 + \Omega_\gamma (1+z')^4}} \approx 3.75 \times 10^5 \text{ years} \ . \tag{21}$$

3. For the abundances we have

$$\Omega_\lambda(z_d) = \Omega_\lambda(z_0), \quad \Omega_m(z_d) = \Omega_m(z_0)(1+z_d)^3, \quad \Omega_\gamma(z_d) = \Omega_\gamma(z_0)(1+z_d)^4 \ , \tag{22}$$

since λ is constant, whereas matter m and radiation γ redshfit as R^{-3} and R^{-4} respectively.