TMP-TC2: Cosmology

Problem Set 4 16, 17, 18 May 2023

1. Comoving Distance and Redshift

The comoving distance χ describes the distance between objects factoring out the expansion of the universe. Therefore, the comoving distance of two comoving objects does not change with cosmological time, while the proper distance $d^p = R(t)\chi$ does.

1. Consider the FLRW metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \right). \tag{1}$$

   How is the comoving distance related to the coordinate distance r?

2. Take an object that emitted photons at time t_1. An observer detects these photons at time t_2. Find an equation for the comoving distance between the emitting object and the observer.

In the Universe we cannot measure distances directly. That's why astronomers rely on light signals from distant objects and measure their redshifts. These redshift measurements provide a powerful tool for studying the large-scale structure and evolution of the Universe. The aim of the next part is to understand the connection between the redshift and scale factor.

3. By considering a comoving emitter of photons and a comoving observer, show that

$$1 + z = \frac{R(t_2)}{R(t_1)} \tag{2}$$

   where t_1 is the time of the emission, t_2 is the time of the observation and z is the redshift parameter defined by $1+z = \frac{\lambda_2}{\lambda_1}$, and as usual λ is the wavelength.

4. Show that a light signal with redshift z was emitted when the age of the Universe was

$$t(z) = \int_z^\infty \frac{\mathrm{d}z'}{(1+z')H} \tag{3}$$

5. As an example consider a matter dominated Universe. One measures the redshift $z = 2$ for a light signal of a comoving galaxy. What time passed since the emission of this signal? How far away was the galaxy at emission? Take the current Hubble constant $H_0 \approx 73 \frac{\text{km}}{\text{s Mpc}}$.

2. Evolution of the Universe

We assume that the present abundances are $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, the temperature of the present cosmic radiation is $T = 2.73$ K, the age of the universe is $t_0 = 14 \cdot 10^9$ years and the Hubble constant 73 km s^{-1}Mpc^{-1}.

1. Find the current vacuum energy density ρ_Λ in $\dfrac{GeV}{cm^3}$ and find the cosmological constant Λ in GeV^2.
2. Why we can neglect the value Ω_γ at the present time ? What is the shape of the universe ?
3. Find the time when $\rho_m = \rho_\Lambda$ and $\rho_m = \rho_\gamma$. Consider the approximate Friedmann equations with only the dominant components involved.
   Hint : For photons we have the equation
   $$\rho = \frac{\pi^2}{30} g T^4$$
   where g is the number of degrees of freedom. (We will derive this formula on the next sheet)

3. Dipole anisotropy of the Cosmic Microwave Background

The Universe is filled with the Cosmic Microwave Background (CMB) radiation. Assume that the Earth is moving inside this medium with velocity $\mathbf{v}$ and show that the temperature changes as

$$T_{OBS} = T_{CMB} \left(\frac{\sqrt{1 - v^2}}{1 + v \cos\theta} \right) .$$

Here θ is the angle between the direction of observation and Earth's velocity. Using the above, estimate v for a dipole anisotropy of the order of 10^{-3}.

4. Photon Decoupling in Numbers

The temperature of the Cosmic Background adiation today is $T_{CMB} = 2.725$ K. Assume that photon decoupling took place at $T_d \approx 0.25$ eV.

1. Calculate the redshift of decoupling z_d. You can assume that the temperature redshifts as $T \sim R^{-1}$.
2. Find the age of the universe t_d at this redshift.
3. Estimate the abundances $\Omega_\Lambda,\ \Omega_m,\ \Omega_\gamma$ at z_d.

Indication: According to the Λ-CDM model, the abundances today are

$$\Omega_\Lambda \approx 0.73 \pm 0.04, \quad \Omega_m \approx 0.27 \pm 0.04, \quad \Omega_\gamma \approx 8.2 \times 10^{-5} .$$