TMP-TC2: Cosmology

Solutions to Problem Set 3

9, 10, 11 May 2023

1. Universe evolutions

The flat and homogeneous universe is described by the Robertson-Walker metric

$$ds^2 = dt^2 - R(t)^2 \left(dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \ .$$

To determine the temporal evolution of the scale factor $R(t)$ we consider the Friedmann equations in the form, which you can get by insulating ρ in the first equation and plug it in the second :

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\lambda}{3} = \frac{8\pi G}{3}\rho,$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \lambda = -8\pi G p.$$

— *The universe composed of radiation and a cosmological constant* $\lambda > 0$

The Friedmann equations become

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{\lambda}{3} = \frac{8\pi G}{3}\rho$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \lambda = -\frac{8\pi G}{3}\rho.$$

We sum the two equations and we obtain the differential equation for $R(t)$.

$$\ddot{R}R + \dot{R}^2 = \frac{2\lambda}{3}R^2 \ .$$

We can write it as

$$\frac{1}{2}\frac{d^2}{dt^2}\left(R^2\right) = \frac{2\lambda}{3}R^2 \ .$$

We define $x \equiv R^2$, and the equation becomes

$$\ddot{x} = \frac{4\lambda}{3}x \ .$$

The general solution is

$$x(t) = A\sinh\left(\sqrt{\frac{4\lambda}{3}}\,t + \varphi_0\right) \ ,$$

where A and φ_0 depend of the initial conditions. Then, the factor scale is

$$R(t) = \sqrt{A}\sinh^{\frac{1}{2}}\left(2\sqrt{\frac{\lambda}{3}}\,t + \varphi_0\right) \ .$$

We use $R(0) = 0$, then $\varphi_0 = 0$ and the final result is [1]

$$R(t) = \sqrt{A}\sinh^{\frac{1}{2}}\left(2\sqrt{\frac{\lambda}{3}}\,t\right) \ .$$

— *The universe composed of matter and a cosmological constant* $\lambda > 0$

As before we combine the Friedmann equations to find a differential equation for $R(t)$.

$$\ddot{R}R + \frac{1}{2}\dot{R}^2 = \frac{\lambda}{2}R^2 \ .$$

To solve it, we can change of variables $R = x^\alpha$. The equation become

$$\ddot{x} + \frac{\dot{x}^2}{x}\left(\frac{3}{2}\alpha - 1\right) = \frac{\lambda}{2\alpha}x.$$

If we choose $\alpha = \frac{2}{3}$, the second term disappear and we find the following equation :

$$\ddot{x} = \frac{3\lambda}{4}x.$$

The general solution is

$$x(t) = A\sinh\left(\sqrt{\frac{3\lambda}{4}}\,t + \varphi_0\right) \ .$$

The final result is :

$$R(t) = A^{\frac{2}{3}}\sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\frac{\lambda}{3}}\,t\right) \ .$$

We can find the age of the universe in function of H_0 and Ω_m. By using the last relation, we find the Hubble constant at present time t_0 :

$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \sqrt{\frac{\lambda}{3}}\coth\left(\frac{3}{2}\sqrt{\frac{\lambda}{3}}\,t_0\right) \ .$$

By inverting this expression [2], we get :

$$t_0 = \frac{2}{3}\sqrt{\frac{3}{\lambda}}\text{arccoth}\left(\sqrt{\frac{3H_0^2}{\lambda}}\right) = \frac{1}{3}\sqrt{\frac{3}{\lambda}}\ln\frac{\sqrt{\frac{3H_0^2}{\lambda}}+1}{\sqrt{\frac{3H_0^2}{\lambda}}-1} = \frac{1}{3}\sqrt{\frac{3}{\lambda}}\ln\frac{1+\sqrt{\frac{\lambda}{3H_0^2}}}{1-\sqrt{\frac{\lambda}{3H_0^2}}} \ .$$

1. Here we describe a universe which is dominated by radiation and the cosmological constant throughout its evolution. In reality, the universe had a phase of domination by radiation and then, a phase of matter domination(non-relativistic). Now the universe seems to be dominated by matter and an energy content, whose properties are very similar to those of a cosmological constant.

2. We use the following relation $\text{arccoth}x = \frac{1}{2}\ln\frac{x+1}{x-1}$ for $|x| > 1$.

We can substitute λ with Ω_m. For this, use the first Friedmann equation at present time t_0.

$$H_0^2 = \frac{\lambda}{3} + \frac{8\pi G}{3}\rho(t_0)\ .$$

We divide by H_0^2 and we find :

$$\frac{\lambda}{3H_0^2} = 1 - \Omega_m\ .$$

We can express t_0 as :

$$\begin{aligned} t_0 &= \frac{1}{3H_0}\frac{1}{\sqrt{1-\Omega_m}}\ln\frac{1+\sqrt{1-\Omega_m}}{1-\sqrt{1-\Omega_m}} \\ &= \frac{1}{3H_0}\frac{1}{\sqrt{1-\Omega_m}}\ln\frac{(1+\sqrt{1-\Omega_m})^2}{\Omega_m} \\ &= \frac{2}{3H_0}\frac{1}{\sqrt{1-\Omega_m}}\ln\frac{1+\sqrt{1-\Omega_m}}{\sqrt{\Omega_m}}\ . \end{aligned}$$

2. The fate of the universe

1. We start from the first Friedmann equation written using abundances

$$\Omega_{\text{mat}} + \Omega_k + \Omega_\lambda = 1.$$

The line $\Omega_\lambda = 1 - \Omega_{\text{mat}}$ defines a flat universe. Above this curve, we have $k = 1$ and $k = -1$ below. To study acceleration, we consider the second Friedmann equation to obtain

$$\frac{\ddot{R}}{R} + \frac{4\pi G}{3}\rho - \frac{\lambda}{3} = 0.$$

By identifying the abundances, we get

$$\Omega_\lambda = \frac{\Omega_{\text{mat}}}{2} + \frac{\ddot{R}}{RH^2}.$$

The curve $\Omega_\lambda = \Omega_{\text{mat}}/2$ describes a universe with a zero acceleration. So we have a universe in acceleration above the curve and deceleration below.

2. In this case, the Friedmann equations can be rewritten as

$$\begin{cases} \dfrac{\ddot{R}}{R} = \dfrac{\lambda}{3}, \\ \dot{R}^2 + k = \dfrac{\lambda}{3}R^2. \end{cases}$$

By solving the first equation, we can obtain constraints on the spatial curvature k.

Case $\lambda > 0$: In this case, the solution is given by

$$R(t) = A\exp\left(\sqrt{\frac{\lambda}{3}}\,t\right) + B\exp\left(-\sqrt{\frac{\lambda}{3}}\,t\right).$$

By inserting this expression in the second equation, we obtain

$$k = \frac{4AB}{3}\,\lambda.$$

This constraint tell us, that for $k = 1$, A and B have to be both positive or negative. Then, an initial singularity is impossible. On the other side, if $k = -1$, A and B have to be of opposite sign, and an initial singularity is possible for $A = -B = \sqrt{3/(4\lambda)}$. Finally, the solution is

$$R(t) = \sqrt{\frac{3}{\lambda}}\sinh\left(\sqrt{\frac{\lambda}{3}}\,t\right).$$

<u>Case $\lambda < 0$:</u> In this case, the solution is given by

$$R(t) = A\cos\left(\sqrt{\frac{|\lambda|}{3}}\,t\right) + B\sin\left(\sqrt{\frac{|\lambda|}{3}}\,t\right).$$

The second equation give :

$$k = \frac{\lambda}{3}\left(A^2 + B^2\right).$$

Since $\lambda < 0$, k have to be -1. Then, for $A = 0$, $B = \sqrt{3/|\lambda|}$, we have an initial singularity. Finally, we get

$$R(t) = \sqrt{\frac{3}{|\lambda|}}\sin\left(\sqrt{\frac{|\lambda|}{3}}\,t\right).$$

3. For the case where we neglect the cosmological constant, we can write the first Friedmann equation :

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho.$$

. Since the energy of the universe is conserved, ρR^3 is constant and we use it to express ρ as a function of R. The we will rephrase the last equation in the new variable $r^2 = (R/R_0)^2$ Using these facts, we find

$$\dot{r}^2 = \frac{8\pi G}{3}\frac{\rho_0}{r} - \frac{k}{R_0^2}$$

Using the definition of the abundances, we write Ω_k in terms of Ω_λ and Ω_{mat} to get

$$\dot{r}^2 = H_0^2\left[\Omega_m^0\left(\frac{1}{r} - 1\right) + 1\right].$$

We can write this equation as :

$$E_{kin} + U(r) = E_{tot},$$

with

$$U(r) = -\frac{H_0^2 \Omega_m^0}{r},$$

and

$$E_{tot} = H_0^2 \left(1 - \Omega_m^0\right).$$

The potential is monotonic, negative and $U(r) \to 0$, when $r \to \infty$. Then, we first deduce, that for all possible values of E_{tot}, there is an initial singularity. Secondly, depending on the sign of E_{tot}, we have an infinite expansion or a collapse in the future. For $E_{tot} > 0$, we have an infinite expansion. Then, $\Omega_m^0 < 1$ is the condition to have an infinite expansion. Otherwise, the universe eventually collapse in the future.

4. For this point, we proceed as before. We write the first Friedmann equation as

$$E_{kin} + U(r) = 0,$$

with

$$U(r) = -\frac{H_0^2}{2}\left[\Omega_m^0\left(\frac{1}{r} - 1\right) + \Omega_\lambda^0\left(r^2 - 1\right) + 1\right].$$

We will study this potential in the cases where $\Omega_\lambda > 0$ and $\Omega_\lambda < 0$ separately.
Case $\Omega_\lambda < 0$: First, remark that $U(r)$ goes from 0, for small r, to ∞ for large r. We compute the derivative :

$$U'(r) = -\frac{H_0^2}{2}\left[2\Omega_\lambda^0 r - \frac{\Omega_m^0}{r^2}\right] > 0.$$

The potential is monotone. Then, there is an initial singularity and the universe grows until r_b, which is define as $U(r_b) = 0$, then it decreases and collapse.
Case $\Omega_\lambda > 0$: In this case, $U(r) \to -\infty$ for both $r \to 0$ and $r \to \infty$. Then, the potential has a maximum. We find it as follow :

$$U'(r_{max}) = -\frac{H_0^2}{2}\left[2\Omega_\lambda^0 r_{max} - \frac{\Omega_m^0}{r_{max}^2}\right] = 0,$$

and we obtain

$$r_{max} = \left(\frac{\Omega_m^0}{2\Omega_\lambda^0}\right)^{1/3}.$$

And we get :

$$U_{max} = U(r_{max}) = -\frac{H_0^2}{2}\left[\frac{3}{2^{2/3}}(\Omega_m^0)^{2/3}(\Omega_\lambda^0)^{1/3} + (1 - \Omega_m^0 - \Omega_\lambda^0)\right].$$

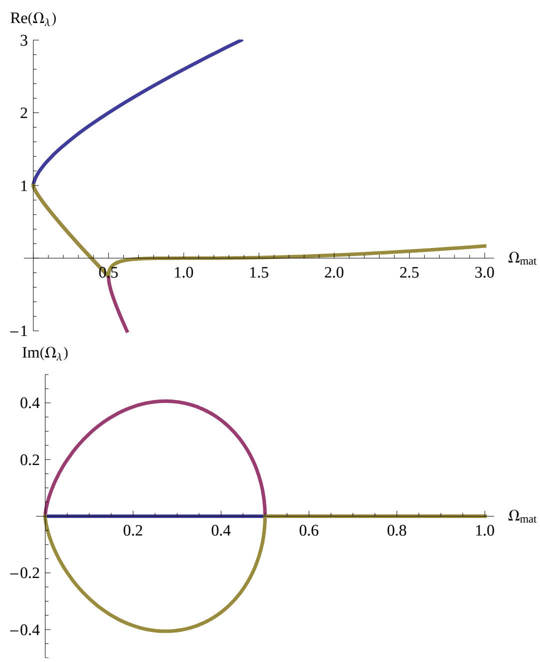
Depending on the sign of U_{max}, we have zero zero or two zeros if $U_{max} < 0$ or $U_{max} > 0$ respectively. To study this inequalities, we find the solutions of the equation $U_{max} = 0$. They are given by

$$\begin{cases} \Omega_\lambda^1 = 1 - \Omega_m + \dfrac{3}{2}\left(\dfrac{\Omega_m^2}{D^{1/3}} + D^{1/3}\right), \\ \Omega_\lambda^2 = 1 - \Omega_m - \dfrac{3}{4}\left(\dfrac{(1+i\sqrt{3})\Omega_m^2}{D^{1/3}} + (1-i\sqrt{3})D^{1/3}\right), \\ \Omega_\lambda^3 = 1 - \Omega_m - \dfrac{3}{4}\left(\dfrac{(1-i\sqrt{3})\Omega_m^2}{D^{1/3}} + (1+i\sqrt{3})D^{1/3}\right) \end{cases}$$

with

$$D = \Omega_m^2 - \Omega_m^3 + \sqrt{\Omega_m^4 - 2\Omega_m^5}.$$

In the following graphs, we plot first the real part of the different solutions and the imaginary part in the second graph.



We conclude from these graphs, that Ω_λ^1(blue curve) is real and positive for all value of Ω_m, then this solution is physic. Ω_λ^2(red curve) is complex for $\Omega_m < 0.5$ and becomes real and negative for $\Omega_m > 0.5$, then this is unphysical. Finally, Ω_λ^3(yellow curve) is complex for $\Omega_m < 0.5$, then it becomes real and negative for $0.5 < \Omega_m < 1$ and it is real and positive for $\Omega_m > 1$, then it is physical from $\Omega_m > 1$.

Above the first curve(Ω_λ^1), we have $U_{max} > 0$ and $r_{max} < 1 = r_0$(today), then there is no initial singularity and the universe expands forever.

Between the two solutions, we have $U_{max} < 0$, then there is an initial singularity and the universe expands forever.

Finally, under the second curve(Ω_λ^3), we have $U_{max} > 0$ and $r_{max} > 1 = r_0$(today), then there is an initial singularity and the universe will collapse in the future.

Remark : Experimentally, it has been measured :

$$\Omega_m = 0.24 \pm 0.04,$$

$$\Omega_\lambda = 0.76 \pm 0.06,$$

$$\Omega_{total} = 1.003 \pm 0.017.$$

The universe has an initial singularity, it will expand forever and is accelerating. It is not known, whether the universe is closed or open, $\Omega_k = -0.003 \pm 0.017$.

3. Recollapsing Universe

1. If we have a recollapsing universe, there is a time when the universe stops its expansion, i.e. $\dot{R} = 0$. Using the first Friedmann equation, this gives

$$R_m = \frac{8\pi G}{3}\rho R^3 = \frac{8\pi G}{3}\rho_m \tag{1}$$

where ρ_m is constant. The mass of the universe is $M = R_m^3 \rho_m$. Using this we obtain

$$R_m = (2GM)^{\frac{1}{4}} \approx 3.4 \cdot 10^{-3} m \tag{2}$$

2. We can rewrite the first Friedmann equation to

$$\frac{\dot{R}^2}{R^2} = \frac{R_m}{R^3} - \frac{1}{R^2}. \tag{3}$$

This yields

$$\dot{R} = \pm\sqrt{\frac{R_m}{R} - 1} \tag{4}$$

Integrating this equation with positive sign leads to

$$t = -R\sqrt{\frac{R_m}{R} - 1} - R_m \arctan(\sqrt{\frac{R_m}{R} - 1}) + R_m\frac{\pi}{2} \tag{5}$$

where we used that $R \to 0$ for $t \to 0$ to find the integration constant. The universe reaches its maximal size for $R = R_m$, which corresponds to $t_m = \frac{\pi}{2}R_m$. After this point we must continue the solution by taking the negative sign in (4). We see that the solution is symmetrical around R_m. Therefore, the total lifetime is

$$t_{\text{life}} = \pi R_m \tag{6}$$

3. The Friedmann equation in that case are

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$
$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}$$

Since we have a negative cosmological constant, $\frac{\ddot{R}}{R} < 0$ which tells us that there was a singularity. Combining the two Friedmann equations leads to

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = \Lambda \tag{7}$$

The solution is

$$R(t) = R_m \cos\left(\frac{\sqrt{3|\Lambda|}}{2}t + B\right)^{\frac{2}{3}} \tag{8}$$

Using the fact that at $t = 0$ was a singularity, gives $B = -\frac{\pi}{2}$. Therefore, the maximal expansion is reached at $t_m = \frac{\pi}{\sqrt{3|\Lambda|}}$ and the lifetime of the universe is

$$t_{\text{life}} = \frac{2\pi}{\sqrt{3|\Lambda|}} \tag{9}$$