TMP-TC2: COSMOLOGY

Solutions to Problem Set 2 2, 3, 4 May 2023

1. Friedmann–Lemaître–Robertson–Walker (FLRW) metric in other coordinate systems

It is known that any homogeneous space with constant spatial curvature can be described by the FLRW metric,

$$ds^2 = dt^2 - R(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right) , \tag{1}$$

with $k = 1, 0, -1$. Metrics which describe the same homogeneous spaces with spatial constant curvature should therefore be obtained by a change of variables from the metric (1). To have the term $d\chi^2$ with coefficient 1, we require that

$$\frac{dr}{\sqrt{1 - kr^2}} = d\chi \ .$$

We can solve these equations for the different value of k. We find

$$\begin{aligned} k = 1 \quad &\rightarrow \quad r = \sin \chi \\ k = 0 \quad &\rightarrow \quad r = \chi \\ k = -1 \quad &\rightarrow \quad r = \sinh \chi \ . \end{aligned}$$

With these changes of coordinates the metric (1) takes exactly the forms given in the exercise. Therefore, the above describe the same type of space as the FLRW metric.

2. Energy-momentum tensor of a perfect fluid

We have seen that the energy-momentum tensor of a perfect fluid in an arbitrary coordinate system is

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g^{\mu\nu} , \tag{2}$$

where ρ is the energy density, p the pressure and u^μ the four-velocity of the medium. The conservation law $\nabla_\nu T^{\mu\nu} = 0$, yields

$$\partial_\nu T^{\mu\nu} + \Gamma^\mu{}_{\nu\lambda} T^{\ \nu\lambda} + \Gamma^\nu{}_{\nu\lambda} T^{\mu\lambda} = 0 \ . \tag{3}$$

Using eq. (2) in the rest frame of the fluid and considering the case $\mu = 0$, the above gives for the spatially flat FLRW metric

$$\begin{aligned}
&\partial_\nu T^{0\nu} + \Gamma^0{}_{\nu\lambda} T^{\nu\lambda} + \Gamma^\nu{}_{\nu\lambda} T^{0\lambda} = 0 \\
\Rightarrow\ &\partial_0 T^{0\nu} + \Gamma^0{}_{00} T^{00} + \Gamma^0{}_{ij} T^{ij} + \Gamma^\nu{}_{\nu 0} T^{00} = 0 \\
\Rightarrow\ &\partial_0 T^{00} + \Gamma^0{}_{ii} T^{ii} + \Gamma^i{}_{i0} T^{00} = 0 \\
\Rightarrow\ &\frac{d\rho}{dt} + 3RR' \frac{1}{R^2} p + 3\frac{R'}{R}\rho = 0 \\
\Rightarrow\ &\frac{d}{dt}(R^3\rho) + p\frac{dR^3}{dt} = 0
\end{aligned} \tag{4}$$

3. Friedmann Equations

1) Here, 'comoving' means that the expansion of the universe is factored out. This means that $u^0 = 1$ and $u^i = 0$. Thus, the energy-momentum tensor is

$$T_{00} = \rho + p - p = \rho \tag{5}$$

$$T_{ii} = -p g_{ii} \tag{6}$$

From the last sheet we know

$$G_{00} = 3\left(\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right) \tag{7}$$

$$G_{ii} = \frac{g_{ii}}{R^2}\left(2R\ddot{R} + \dot{R}^2 + k\right) \tag{8}$$

Taking the Einstein equation $G_{\mu\nu} = 8\pi G T_\mu \nu$ gives for the 00-component

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{9}$$

and for the ii-components we obtain

$$\frac{g_{ii}}{R^2}\left(2R\ddot{R} + \dot{R}^2 + k\right) = -8\pi G p g_{ii}$$

$$2\frac{\ddot{R}}{R} + H^2 + \frac{k}{R^2} = -8\pi G p \tag{10}$$

Using the equation (9), this becomes

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \tag{11}$$

2) First of all we bring the equation that we obtained in the previous exercise into a more familiar form :

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{12}$$

Taking the time derivative of the first Friedmann equation (9) gives

$$2H\dot{H} - 2\frac{k}{R^3}\dot{R} = \frac{8\pi G}{3}\dot{\rho}$$
$$H\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2}\right) = \frac{4\pi G}{3}\dot{\rho}$$

Insert the second Friedmann equation (11) yields

$$-\frac{4\pi G}{3}H(\rho + 3p) - H^3 - \frac{k}{R^2}H = \frac{4\pi G}{3}r\dot{h}o$$

And now inserting the first Friedmann equation results in the desired equation :

$$-\frac{4\pi G}{3}H(\rho + 3p) - \frac{8\pi G}{3}H\rho = \frac{4\pi G}{3}\dot{\rho}$$
$$\dot{\rho} + 3H(\rho + p) = 0$$

4. General equation of state

1. Solving Eq.(4) gives,
$$\rho(R) \sim \frac{1}{R^{3(1+w)}}. \tag{13}$$
2. Solving Friedmann equation gives,
$$R(t) \sim t^{\alpha}, \quad \alpha = \frac{2}{3}\frac{1}{1+w}, \tag{14}$$
from which we have,
$$\rho(t) \sim \frac{1}{t^2}. \tag{15}$$
As $t \to 0$, both $\rho(t)$ diverge and $R \to 0$.
3. Differentiating Eq.(14), we have,
$$\ddot{R}(t) \sim \alpha(\alpha - 1)t^{\alpha-2}. \tag{16}$$
The Universe expands with acceleration if $\alpha - 1 > 0$, or if $-1 < w < -\frac{1}{3}$.

5. Einstein Universe

1. The first Friedmann equation gets immediately
$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{17}$$

From the Einstein equation we obtain for the (ii)-components

$$\frac{g_{ii}}{R^2}(2R\ddot{R} + \dot{R}^2 + k) - g_{ii}\Lambda = -8\pi G p g_{ii}$$

$$2\frac{\ddot{R}}{R} + H^2 + \frac{k}{R^2} = -8\pi G p + \Lambda$$

Using (17) we obtain

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{18}$$

2. The Einstein universe is static which means that $\dot{R} = 0$ and $\ddot{R} = 0$. Furthermore we assume a matter dominated universe, i.e. $p \approx 0$. Inserting this into the second Friedmann equation gives

$$\Lambda = 4\pi G\rho \tag{19}$$

3. Inserting the previous result into the first Friedmann equation yields

$$R = \sqrt{\frac{k}{\Lambda}}$$

We observe that k can only be $+1$ and thus the shape of the universe is a closed 3-sphere.

4. First we restore the speed of light for the numerical estimations.

$$R_0 = \frac{1}{\sqrt{\Lambda}} = \frac{c}{\sqrt{4\pi G\rho}},$$

For the cosmological constant, we have

$$\Lambda = \frac{1}{R_0^2} = 3.08 \times 10^{-53}[\mathrm{m}^{-2}] = 1.19 \times 10^{-84}[\mathrm{GeV}^2].$$

To obtain the last equality, we multiplied by $\hbar^2 c^2$ and transformed J in GeV with $1\mathrm{GeV} = 1.6 \times 10^{-10}[\mathrm{J}]$. For the matter density :

$$\rho = \frac{c^2}{4\pi R^2 G}\ [\mathrm{kg/m^3}] = 6.30 \times 10^{-27}[\mathrm{kg/m^3}] = 2.70 \times 10^{-47}[\mathrm{GeV}^4].$$

For the last equality, we multiplied by $\hbar^3 c^5$ and transformed J in GeV.