TMP-TC2: COSMOLOGY

Problem Set 2 2, 3, 4 May 2023

1. Friedmann–Lemaître–Robertson–Walker (FLRW) metric in other coordinate systems

Show that the following line elements correspond to homogeneous and isotropic space with constant spatial curvature :

$$\begin{aligned}
ds^2 &= dt^2 - R(t)^2 \left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \\
ds^2 &= dt^2 - R(t)^2 \left(d\chi^2 + \chi^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \\
ds^2 &= dt^2 - R(t)^2 \left(d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \ .
\end{aligned}$$

2. Energy-momentum tensor of a perfect fluid

Starting from the conservation of the energy-momentum tensor of a perfect fluid in arbitrary coordinate systems, show that for the flat FLRW metric

$$\frac{d}{dt}(\rho R^3) + p\frac{d}{dt}R^3 = 0. \tag{1}$$

3. Friedmann Equations

Consider a homogeneous and isotropic universe described by the FLRW metric. Assume that the universe is filled with a comoving perfect fluid with energy ρ and pressure p.

1. Starting from the Einstein equation, derive the Friedmann equations

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3}$$

   with $H = \frac{\dot{R}}{R}$

2. In the previous problem you found that the conservation of the energy-momentum tensor of a perfect fluid implies equation (1). Derive the same equation from the Friedmann equations.

4. General equation of state

Consider the Universe filled with matter with the following equation of state

$$p = w\rho, \quad w = \texttt{Const.} \tag{4}$$

Assume for simplicity $k = 0$.

1. From Eq.(1) determine the dependence $\rho = \rho(R)$.
2. From Friedmann equation find $R(t)$ and $\rho(t)$ for $w > -\frac{1}{3}$. How do they behave at $t \to 0$?
3. Find which values of w correspond to the accelerating expansion of the Universe (i.e. $\ddot{R} > 0$).

5. Einstein Universe

Albert Einstein initially assumed that the universe was static and had a uniform distribution of matter, which led him to introduce the cosmological constant Λ in his field equations :

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5}$$

1. What modifications occur in the Friedmann equations with the inclusion of the cosmological constant ?
2. Now assume the universe to be static and dominated by matter. Find an expression for Λ in terms of the energy density ρ.
3. How does R depends on Λ ? What is the shape of the universe ?
4. Assuming the radius of the universe to be $R_0 \sim 1.38 \times 10^{10}$ l.y., find the matter density ρ and the cosmological constant λ required in maintaining the universe static. Give answers in units MKSA and GeV.