TMP-TC2: Cosmology

Solutions to Problem Set 10 27, 28, 29 June 2023

1 Flatness Problem

Using the given equation $\Omega - 1 = \frac{k}{R^2 H^2}$ we can find for an arbitrary time t :

$$|\Omega(t) - 1| = \frac{R_0^2 H_0^2}{R(t)^2 H(t)^2} |\Omega_0 - 1| \tag{1}$$

With $R(t) \propto t^n$ ($n = \frac{1}{2}$ for radiation domination and $n = \frac{2}{3}$ for matter domination) we get

$$|\Omega(t) - 1| = \left(\frac{t}{t_0}\right)^{2(1-n)} |\Omega_0 - 1| \tag{2}$$

Inserting the time of recombination $t_R \approx 3.7 \cdot 10^5$ years, we obtain

$$|\Omega(t_R) - 1| \approx \mathcal{O}(10^{-8} - 10^{-9}) \tag{3}$$

It seems that this number is very fine-tuned and surprisingly close to the value zero corresponding to a flat universe. But why ? This is the flatness problem.

2 Horizon Problem

Let us assume that at one point in the past, a signal was emitted. Then the proper distance between the observer and the source is given at time t_0 by

$$d(t_0) = R(t_0) \int_{t_e}^{t_0} \frac{1}{R(t)} \mathrm{d}t \tag{4}$$

If t_e is the time of the emission of the CMB and t_0 is the age of the universe today, then d describes the distance between us and the CMB.
The size of the causally connected region at t_e is

$$D(t_0) = R(t_0) \int_{0}^{t_e} \frac{1}{R(t)} \mathrm{d}t \tag{5}$$

Then the angle that contains one causally connected region in the sky is

$$\theta = 2 \arctan\left(\frac{1}{2} \frac{D(t_0)}{d(t_0)}\right) \tag{6}$$

For a matter dominated universe we have $R(t) \propto t^{\frac{2}{3}}$. Therefore, we obtain for D and d

$$D(t_0) = 3 t_0^{\frac{2}{3}} t_e^{\frac{1}{3}} \tag{7}$$

$$d(t_0) = 3 t_0^{\frac{2}{3}} (t_0^{\frac{1}{3}} - t_e^{\frac{1}{3}}) \tag{8}$$

With $1+z=\frac{R(t_0)}{R(t_e)}=\left(\frac{t_0}{t_e}\right)^{\frac{2}{3}}$ we obtain for the angle

$$\theta = 2\arctan\left(\frac{1}{2}\frac{1}{\sqrt{1+z}-1}\right) \tag{9}$$

With $z \approx 1500$, we get the angle $\theta \approx 1.52°$.
The problem with this small angle is that in the CMB are many causally disconnected patches. However, the CMB is very isotropic. How can this be? One solution to this is for example inflation. We will discuss this on the next sheet.

3 Phase Transitions and Bubble Nucleation

1. Above the critical temperature the potential has exactly one minimum at $\phi = 0$. So the vacuum expectation value of the whole space is zero which means that there is a non-broken Z_2 symmetry $\phi \mapsto -\phi$ everywhere. As soon as the temperature decreases to a value below the critical temperature, the potential obtains two minima. Now each causally disconnected region has to choose one of these two vacuum expectation values. Hence, the Z_2 symmetry is broken.
   During this phase transition, it can happen that domain walls appear. We will analyze them for another potential below.
2. The behaviour of the potential is plotted in the figure 1. We can observe that there is only one minimum for very high temperatures. With decreasing temperatures, there appear additional minima for $|\phi| \neq 0$. First, these new minima are above the level of the minimum at zero, but below a certain critical temperature, these new minima become to be at a lower level. At the time around the critical temperature, the phase transition happens.

3. The static field equation is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \frac{\partial V}{\partial \phi} = 0 \tag{10}$$

   Multiplying this equation with $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ and applying the chain rule gives

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial V}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}x}$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\right) = \frac{\mathrm{d}V}{\mathrm{d}x}$$

   Integrating this equation and assuming that the potential vanishes for $x \to \infty$ yields the Bogomolny equation

$$\phi'(x) = \pm\sqrt{2V(\phi)} \tag{11}$$

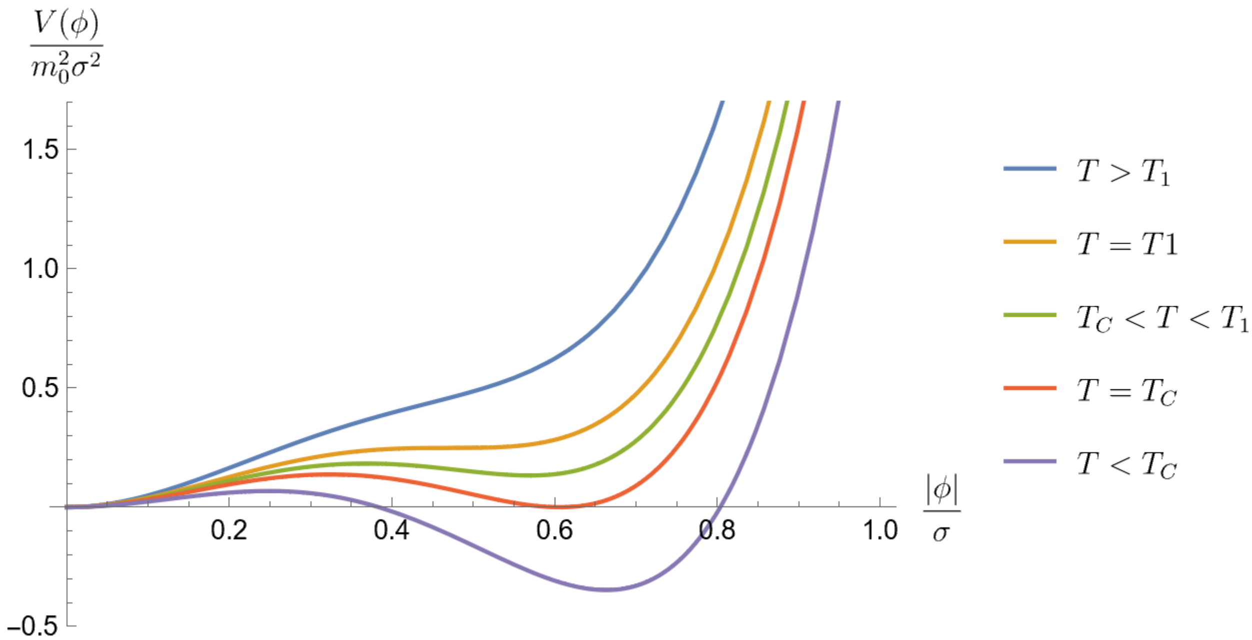

FIGURE 1 – Coleman-Weinberg potential for different temperatures

Now we can insert the potential and integrate

$$x = \int \mathrm{d}\phi \frac{1}{\sqrt{2\lambda}(\phi^2 - v^2)\phi}$$
$$= \frac{1}{\sqrt{2\lambda}v^2} \left(-\ln\phi + \frac{1}{2}\ln\bigl(\phi^2 - v^2\bigr) \right) + \text{const}$$

Solving this for ϕ gives

$$\phi(x) = \frac{\pm v}{\sqrt{1 - e^{2\sqrt{2\lambda}v^2(x-x_0)}}} \tag{12}$$

This is the domain wall solution that separates two regions with different vacuum expectation values. Here we have the symmetric region for $\phi = 0$ and the non-symmetric region for $\phi = \pm v$.
Similarly one can find

$$\phi(x) = \frac{\pm v}{\sqrt{1 - e^{-2\sqrt{2\lambda}v^2(x-x_0)}}} \tag{13}$$

The solutions are plotted in figure 2.
The energy can be calculated as follows

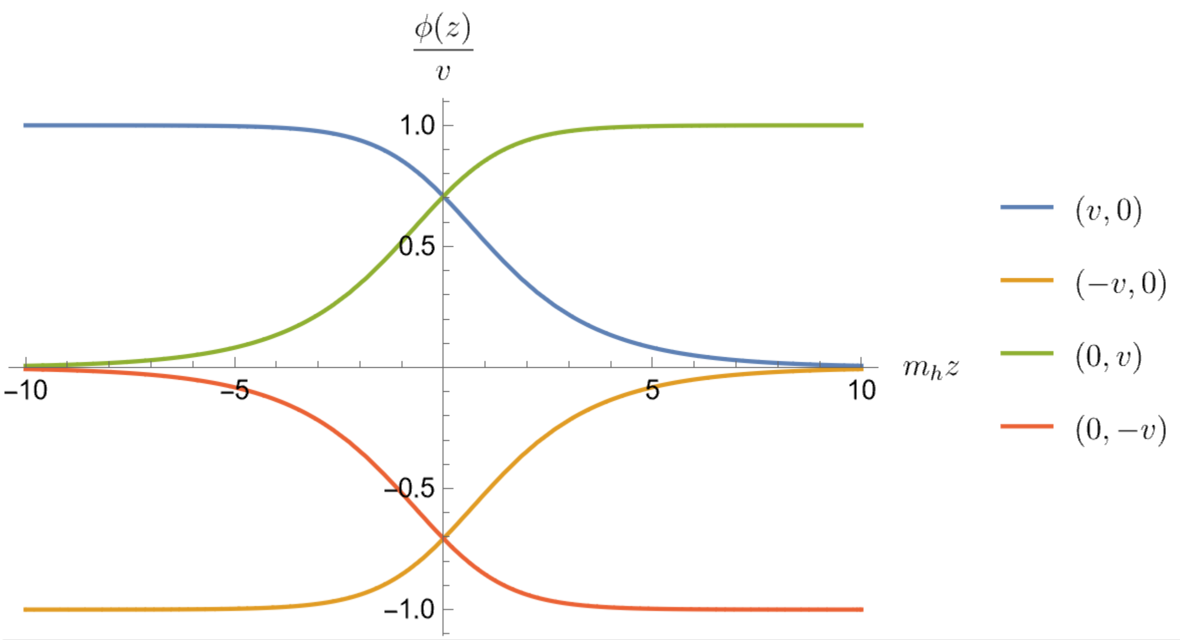

FIGURE 2 – Domain wall solutions for the ϕ^6 model.

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \mathrm{d}x (\frac{1}{2}\phi'^2 + V) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x 2V \\
&= \int_{\pm v}^{0} \frac{1}{\phi'} \mathrm{d}\phi 2V \\
&= \int_{\pm v}^{0} \mathrm{d}\phi \sqrt{2V} \\
&= \int_{\pm v}^{0} \mathrm{d}\phi \sqrt{2\lambda}\phi(\phi^2 - v^2) \\
&= \frac{\sqrt{2\lambda} v^4}{4}
\end{aligned}
$$

4. The energy of a domain wall at the initial time is given by

$$E_0 = \sigma_{\mathrm{DW}} 2\pi R_0 \tag{14}$$

Through the tension of the DW, this bubble is not stable at will collapse. So in general we write the energy for a collapsing bubble configuration to be

$$E = \gamma(t) \sigma_{\mathrm{DW}} 2\pi R(t) \tag{15}$$

where *gamma* is the Lorentz factor. Since energy is conserved we have $E = E_0$ and so we obtain the equation

$$R_0 = \gamma(t) R(t) \tag{16}$$

Inserting *gamma* gives

$$1 - (\frac{\mathrm{d}R}{\mathrm{d}t})^2 = (\frac{R}{R_0})^2 \tag{17}$$

The solutions to this equation are given by

$$R(t) = R_0 \cos\left(\frac{t}{R_0}\right) \tag{18}$$

Hence, we can see that a DW bubble will collapse in this scenario.

5. We can add a term proportional to ϕ^2 such that the minimum at $\phi = 0$ of the ϕ^6 potential gets raised. This means that energetically the broken vacuum is preferred. This leads to pressure on the DW bubble such that it can grow if this pressure is stronger than the tension of the bubble. Therefore, above a critical size, such a bubble will always grow.
6. At some point when the bubbles expand they will collide with each other. At these collisions, a lot of gravitational waves get emitted. This can have a lot of implications on the cosmic microwave background.
   Furthermore, besides domain walls, there are also other types of topological defects that can form. These are for example magnetic monopoles or cosmic string. During the interaction between all these defects, a lot of radiation (gravitational and electromagnetic) can be produced which may again have implications for the CMB.

4 Magnetic Monopole Problem

The energy density of monopoles today is given by

$$\epsilon_{\mathrm{M}}^0 = m_{\mathrm{M}} n_{\mathrm{M}}^0 \ , \tag{19}$$

with $m_{\mathrm{M}} \simeq 10^{17}$ GeV and n_{M}^0 the number density of monopoles today.

No annihilation or creation of monopoles took place, therefore,

$$n_{\mathrm{M}}^0 = n_{\mathrm{M}} R^3 = n_{\mathrm{M}} \left(\frac{T_0}{T_{\mathrm{GUT}}}\right)^3 \ , \tag{20}$$

where n_{M} is their number density at T_{GUT}. Since there is one monopole per Hubble patch,

$$n_{\mathrm{M}} = \frac{1}{r_H^3} = \left(\frac{T_{\mathrm{GUT}}^2}{M_{\mathrm{Pl}}}\right)^3 \ . \tag{21}$$

From the above it is easy to see that

$$n_{\mathrm{M}}^0 = \left(\frac{T_{\mathrm{GUT}}}{M_{\mathrm{Pl}}}\right)^3 T_0^3 \ , \tag{22}$$

which in turn results into

$$\epsilon_{\mathrm{M}}^0 = m_M \left(\frac{T_{\mathrm{GUT}}}{M_{\mathrm{Pl}}}\right)^3 T_0^3 \sim 10^{-14} \frac{g}{cm^3} \ . \tag{23}$$

This energy density is 16 orders of magnitude bigger than the critical energy density $\varepsilon_c \sim 10^{-30}\frac{g}{cm^3}$, so it is in complete disagreement with observations of our Universe today. This problem is called the magnetic monopole problem.
On the next sheet, we will introduce inflation, which would solve the problem, because through an extreme expansion of the universe, the density of magnetic monopoles would decrease to appropriate levels. Besides inflation, there are also other solutions to this problem. If you are interested in it you can have a look on the Langacker Pi mechanism, symmetry non-restoration or the monopole erasure by domain wall collisions.