TMP-TC2: COSMOLOGY

Problem Set 1 25, 26, 27 April 2023

1. Covariant Derivative

1. Remember that a vector transforms under a coordinate transformation from x^μ to $\bar{x}^\mu$ as
$$V^\mu \mapsto \bar{V}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} V^\nu . \tag{1}$$
How does the derivative $\partial_\mu V^\nu$ transform?
Is $\partial_\mu V^\mu = 0$ a coordinate-independent expression?

2. Take the covariant derivative ∇_μ and find how $\nabla_\mu V^\nu$ transforms.
Is $\nabla_\mu V^\mu = 0$ a coordinate-independent expression?

   Hint: Remember that the covariant derivative acts on a vector as
$$\nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} V^\lambda \ ,$$
with $\Gamma^\nu_{\mu\lambda}$ the Christoffel symbols.

3. Using the fact that covariant derivatives obey the Leibniz rule, symbolically $\nabla(AB) = (\nabla A)B + A(\nabla B)$, deduce explicit expressions for $\nabla_\lambda T^{\mu\nu}$, $\nabla_\lambda T_{\mu\nu}$ and $\nabla_\lambda T_\mu{}^\nu$.

2. Metric for a 3-sphere and a 4-dimensional hyperboloid

1. Consider a 3-sphere given by
$$x^2 + y^2 + z^2 + w^2 = 1 \tag{2}$$
Using this constraint, eliminate the w-coordinate in the following metric for a 4-dimensional space:
$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{3}$$

2. Take the coordinates

$$\begin{aligned} x &= \sin\chi\cos\phi\sin\theta \\ y &= \sin\chi\sin\phi\sin\theta \\ z &= \sin\chi\cos\theta \\ w &= \cos\chi \end{aligned}$$

and find the metric in this coordinate system.

3. Now consider a hyperboloid given by

$$x^2 + y^2 + z^2 - w^2 = -1 \tag{4}$$

Eliminate again the w-coordinate in the following metric for a 4-dimensional space:

$$ds^2 = dx^2 + dy^2 + dz^2 - dw^2 \tag{5}$$

4. Take the coordinates

$$\begin{aligned} x &= \sinh\chi\cos\phi\sin\theta \\ y &= \sinh\chi\sin\phi\sin\theta \\ z &= \sinh\chi\cos\theta \\ w &= \cosh\chi \end{aligned}$$

and find the metric in this coordinate system.

3. Friedmann–Lemaître–Robertson–Walker (FLRW) metric

A homogeneous and isotropic universe can be described by the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right] .$$

Note that $k = \pm 1$ or 0. The above for $k = 0$ can also be written as

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] .$$

For $k = 0$ and $k \neq 0$, effectuate the following steps:
1) Write $g_{\mu\nu}$ and determine $g^{\mu\nu}$.
2) Derive the geodesic equations for a particle in this space from its action

$$S = m\int dp\; g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \quad \text{where} \quad \dot{x}^\mu = \frac{d}{dp}x^\mu$$

3) Deduce by identification the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ by writing the equation of motion as $\ddot{x}^{\lambda} = -\Gamma^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$. Verify some results with the usual formula:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_{\mu}g_{\nu\kappa} + \partial_{\nu}g_{\mu\kappa} - \partial_{\kappa}g_{\mu\nu}\right) \quad \text{where} \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

4) Calculate the Riemann tensor using

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\sigma} - (\rho \leftrightarrow \sigma)$$

5) Calculate the Ricci tensor $R_{\mu\nu}$.
6) Determinate the scalar curvature R.
7) Calculate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

4. Volume in curved spacetime

For the FLRW metric with positive curvature, calculate the volume of the spacetime.

Hint: The volume is given by: $V = \int d^3x\sqrt{\gamma}$, with γ the determinant of the spatial part of the metric.