Sheet 7:

Hand-out: Friday, Jun 09, 2023

Problem 1 Generating functionals

In this problem we derive the relation between the free-particle S-matrix and the generating functional containing the Green's function:

$$S[\eta^*,\eta] \equiv \langle 0|\mathcal{T}\ \exp\left[-i\int_{-\infty}^{\infty} dt\ \left(\eta^*(t)\hat{\psi}(t)+\hat{\psi}^\dagger(t)\eta(t)\right)\right]|0\rangle = $$
$$= \exp\left[-i\int_{-\infty}^{\infty} dt\ \eta^*(t)G(t-t')\eta(t)\right]. \tag{1}$$

Here, $\hat{\psi}$ is a bosonic or fermionic field operator and $\eta(t)$ is a time-dependent $\mathbb{C}$ or Grassman number, respectively.

(3.a) Start with the case where $\hat{\psi}(t)=\hat{a}(t)=e^{-i\omega t}\hat{a}$ is a bosonic field in the interaction picture as introduced in the lecture. First, introduce N small time steps $\Delta\tau = 2\tau/N$ to write out the time-ordered exponential in the first line of Eq. (1), where the integration limits are from $-\tau$ to τ, and later $N\to\infty$, $\tau\to\infty$.

(3.b) In (3.a) you obtain a product over many exponentials. Factorize each exponential into parts containing only $\hat{a}$ and $\hat{a}^\dagger$ operators, respectively.

Hint: For $\hat{A}$ and $\hat{B}$ with $\left[[\hat{A},\hat{B}],\hat{A}\right]=\left[[\hat{A},\hat{B}],\hat{B}\right]=0$, it holds:

$$\exp\left[\hat{A}+\hat{B}\right]=\exp\left[\hat{B}\right]\exp\left[\hat{A}\right]\exp\left[[\hat{A},\hat{B}]/2\right] \tag{2}$$

(3.c) Use the result from (3.b) to normal-order the expression. This will allow you to evaluate $S[\eta^*(t),\eta(t)]$ explicitly.

(3.d) Compare your result in (3.c) with the second line of Eq. (1) to show the equality of both expressions in Eq. (1). You may use that for free bosons

$$G(t-t') = -i\theta(t-t')e^{-i\omega(t-t')}. \tag{3}$$

(3.e) Repeat (3.a) - (3.d) for the fermionic driven oscillator with $\hat{\psi}(t)=\hat{c}(t)=e^{-i\epsilon t}\hat{c}$ and Grassman numbers $\eta(t)$!

Hint: Do the calculation separately for $\epsilon>0$ (particles) and $\epsilon<0$ (holes), where in the latter case the ground state is not $|0\rangle$, but $|\psi_0\rangle = c^\dagger|0\rangle$ (hole vacuum). Thus, in 3(c) use anti-normal-order for $\epsilon<0$.

Problem 2 Fermionic coherent states

In this problem we show some basic properties of fermionic coherent states. We will denote by c and c^* the Grassman variables corresponding to the set of fermionic operator $\hat{c}$ and $\hat{c}^\dagger$.

(1.a) Show that the overlap of two coherent states is

$$\langle c^*|c\rangle = e^{c^* c}. \tag{4}$$

(1.b) Show the completeness relation,

$$\int dc^* dc \; |c\rangle\langle c^*| e^{-c^* c} = 1. \tag{5}$$

Hint: Start from the left hand side and use the explicit representation $|c\rangle = |0\rangle + |1\rangle c$.

(1.c) Show the trace formula:

$$\mathrm{tr}\; \hat{A} = \int dc^* dc \; e^{-c^* c} \langle -c^*|\hat{A}|c\rangle. \tag{6}$$

Hint: Show first that: $\delta_{n,m} = \langle n|m\rangle = \int dc^* dc \; e^{-c^* c}\langle -c^*|m\rangle\langle n|c\rangle$, for $n, m = 0, 1$ labeling Fock states.