Sheet 6:

Hand-out: Friday, Jun 02, 2023

Problem 1 The Cooper pair wavefunction

In this problem we derive Cooper's expression for the binding energy of a single Cooper pair. Consider the following Hamiltonian,

$$\hat{\mathcal{H}} = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} \; \hat{c}^\dagger_{\boldsymbol{k},\sigma} \hat{c}_{\boldsymbol{k},\sigma} + \hat{\mathcal{H}}_{\text{int}} \tag{1}$$

as discussed in the lecture.

(2.a) Start from a Fermi-sea $|\text{FS}\rangle$ and make Cooper's ansatz for a state with two more electrons,

$$|\Psi\rangle = \hat{\Lambda}^\dagger |\text{FS}\rangle \qquad \hat{\Lambda}^\dagger = \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}} \; \hat{c}^\dagger_{\boldsymbol{k},\downarrow} \hat{c}^\dagger_{-\boldsymbol{k},\downarrow}. \tag{2}$$

Show that (k_F is the Fermi momentum):

$$|\Psi\rangle = \sum_{|\boldsymbol{k}|>k_F} \phi_{\boldsymbol{k}} \; |\boldsymbol{k}_P\rangle, \qquad \text{with} \quad |\boldsymbol{k}_P\rangle = \hat{c}^\dagger_{\boldsymbol{k},\downarrow} \hat{c}^\dagger_{-\boldsymbol{k},\downarrow} \; |\text{FS}\rangle. \tag{3}$$

In the following exercises we will assume that the Fermi energy $\epsilon_F = \epsilon(k_F) = 0$.

(2.b) Assume that $|\Psi\rangle$ is an eigenstate of $\hat{\mathcal{H}}$, i.e. $\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle$. By comparing components of this vector equation on both sides, show that

$$E\phi_{\boldsymbol{k}} = 2\varepsilon_{\boldsymbol{k}} \; \phi_{\boldsymbol{k}} + \sum_{|\boldsymbol{k}'|>k_F} \langle \boldsymbol{k}_P | \hat{\mathcal{H}}_{\text{int}} | \boldsymbol{k}'_P \rangle \; \phi_{\boldsymbol{k}'} \tag{4}$$

(2.c) Simplify the interaction by making Cooper's seminal ansatz,

$$V_{\boldsymbol{k},\boldsymbol{k}'} \equiv \langle \boldsymbol{k}_P | \hat{\mathcal{H}}_{\text{int}} | \boldsymbol{k}'_P \rangle = \begin{cases} -g_0/V & |\varepsilon_{\boldsymbol{k}}|, |\varepsilon_{\boldsymbol{k}'}| < \omega_{\text{D}} \\ 0 & \text{else} \end{cases} \tag{5}$$

Here ω_{D} describes a narrow energy shell and $V = L^d$ denotes the system's volume. Using this simplified interaction, show that Eq. (4) becomes:

$$\phi_{\boldsymbol{k}} = -\frac{g_0/V}{E - 2\varepsilon_{\boldsymbol{k}}} \sum_{0<\varepsilon_{\boldsymbol{k}'}<\omega_{\text{D}}} \phi_{\boldsymbol{k}'}. \tag{6}$$

(2.d) From Eq. (6) derive a self-consistency equation for the energy E of the Cooper pair! Take the continuum limit by replacing $\frac{1}{V}\sum_{0<\varepsilon_{\boldsymbol{k}}} \to N(0)\int_0^{\omega_{\mathrm{D}}} d\varepsilon$, where $N(0)$ is the density of states per spin per unit volume at the Fermi energy, and show that:

$$1 = g_0 N(0) \int_0^{\omega_{\mathrm{D}}} d\varepsilon \, \frac{1}{2\varepsilon - E} \tag{7}$$

(2.e) Solve Eq. (7) for E, by assuming $2\omega_{\mathrm{D}} - E \approx 2\omega_{\mathrm{D}}$. Show that:

$$E = -2\omega_{\mathrm{D}} \, e^{-\frac{2}{g_0 N(0)}}. \tag{8}$$

Problem 2 Green's functions

In this problem we calculate some important Green's functions which we saw in the lecture.

(1.a) For a bosonic field $\hat{\phi}_{\boldsymbol{q}} = \sqrt{\hbar/(2m\omega_{\boldsymbol{q}})}\left(\hat{a}_{\boldsymbol{q}} + \hat{a}^\dagger_{-\boldsymbol{q}}\right)$ and a Hamiltonian $\hat{\mathcal{H}}_0 = \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}}\left(\hat{a}^\dagger_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}} + 1/2\right)$, show that

$$D(\boldsymbol{q}, t) \equiv -i\langle 0|\mathcal{T}\hat{\phi}_{\boldsymbol{q}}(t)\hat{\phi}_{-\boldsymbol{q}}(0)|0\rangle = -i\frac{\hbar}{2m\omega_{\boldsymbol{q}}}\left[\theta(t)e^{-i\omega_{\boldsymbol{q}}t} + \theta(-t)e^{i\omega_{\boldsymbol{q}}t}\right], \tag{9}$$

and

$$D(\boldsymbol{q}, \nu) = \frac{\hbar}{2m\omega_{\boldsymbol{q}}}\left[\frac{1}{\nu - (\omega_{\boldsymbol{q}} - i0^+)} + \frac{1}{-\nu - (\omega_{\boldsymbol{q}} - i0^+)}\right]. \tag{10}$$

(1.b) For a fermionic field $\hat{c}_{\boldsymbol{k},\sigma}$ and a Hamiltonian $\hat{\mathcal{H}}_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma}$ with ground state $|\psi_0\rangle = \prod_{\sigma,|\mathbf{k}|<k_F} \hat{c}^\dagger_{\mathbf{k},\sigma}|0\rangle$, show that

$$G_{\sigma,\sigma'}(\boldsymbol{k}, \boldsymbol{k}'; t) \equiv -i\langle\psi_0|\mathcal{T}\hat{c}_{\boldsymbol{k},\sigma}(t)\hat{c}^\dagger_{\boldsymbol{k}',\sigma'}(0)|\psi_0\rangle = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\sigma,\sigma'}\begin{cases} -i\theta(|\boldsymbol{k}| - k_F)e^{-i\varepsilon_{\boldsymbol{k}}t} & t > 0 \\ i\theta(k_F - |\boldsymbol{k}|)e^{-i\varepsilon_{\boldsymbol{k}}t} & t < 0 \end{cases} \tag{11}$$

and

$$G(\boldsymbol{k}, \omega) = \frac{1}{\omega - \varepsilon_{\boldsymbol{k}} + i0^+\mathrm{sgn}(\varepsilon_{\boldsymbol{k}})}. \tag{12}$$

Here k_F denotes the Fermi momentum.

Problem 3 Using Grassman integrals

In this exercise, we use Grassman integrals to prove the following identity:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left[A - BD^{-1}C\right] \; \det D, \tag{13}$$

for square matrices A, D of size $N \times N$ and $M \times M$ respectively; B and C are matrices of corresponding sizes. To this end, recall first that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1}^{N} d\alpha_j^* d\alpha_j \prod_{k=1}^{M} d\beta_k^* d\beta_k \; \exp\left[(\alpha^*, \beta^*)\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right], \tag{14}$$

with vectors of Grassman numbers $\alpha, \alpha^*, \beta, \beta^*$ of lengths N, N, M, M, respectively.

(2.a) Separate the expression Eq (14) into an inner and an outer integral, by writing

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1}^{N} d\alpha_j^* d\alpha_j \; \exp[-\alpha^* A\alpha] \; Y[\alpha^*, \alpha], \tag{15}$$

and find an expression for $Y[\alpha^*, \alpha]$ as a Grassman integral over β^* and β.

(2.b) Solve the inner integral and show that its result is given by

$$Y[\alpha^*, \alpha] = \det(D) \;\; \exp\left[\alpha^* B D^{-1} C \alpha\right]. \tag{16}$$

Hint: Use the following Gaussian Grassman integral:

$$\int \prod_j d\eta_j^* d\eta_j \;\; \exp\left[-\eta^* A\eta + j^*\eta + \eta^* j\right] = \det(A) \;\; \exp[j^* A^{-1} j], \tag{17}$$

for matrix A and vectors of Grassman numbers j and j^*.

(2.c) Use the result from (2.b) to solve the outer integral in (2.a). This way, show the identity Eq. (13).