Sheet 3:

Hand-out: Friday, May. 5, 2023

Problem 1 Bogoliubov theory of the Bose-Hubbard model

In this problem, we extend the Bogoliubov theory of weakly interacting bosons to a BEC in a lattice model. You can follow closely the continuum calculation from the lecture. Specifically, consider the Bose-Hubbard Hamiltonian in 3D,

$$\hat{\mathcal{H}} = -t \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left(\hat{a}^\dagger_{\boldsymbol{i}} \hat{a}_{\boldsymbol{j}} + \text{h.c.} \right) + \frac{U}{2} \sum_{\boldsymbol{j}} (\hat{a}^\dagger_{\boldsymbol{j}} \hat{a}_{\boldsymbol{j}} - 1) \hat{a}^\dagger_{\boldsymbol{j}} \hat{a}_{\boldsymbol{j}}, \tag{1}$$

where $[\hat{a}_{\boldsymbol{i}}, \hat{a}^\dagger_{\boldsymbol{j}}] = \delta_{\boldsymbol{i},\boldsymbol{j}}$ are bosonic operators.

(1.a) Determine the normal-ordered Hamiltonian $:\hat{\mathcal{H}}:$ and show that $:\hat{\mathcal{H}}: = \hat{\mathcal{H}}$. Explain the physical meaning of the different terms in the Hamiltonian.

(1.b) Introduce discrete momentum modes, associated with second quantized bosonic operators $\hat{a}_{\boldsymbol{k}}$, and make a variational ansatz which describes a macroscopic occupation of the $\boldsymbol{k} = 0$ mode.

(1.c) Derive the effective Hamiltonian, quadratic in $\hat{a}_{\boldsymbol{k}\neq 0}$, describing low-energy collective excitations.

(1.d) Diagonalize the effective Hamiltonian from (1.c) and derive the Bogoliubov dispersion relation in a lattice.

Problem 2 Classical field theory

In this problem we discuss *classical field theories* and their formulation using Lagrangian densities. Later in the lecture we focus on *quantum field theories*, but it is often useful to relate them (in some limits) to simpler classical theories.

A classical field $\phi_n(\boldsymbol{r}, t)$ with components n is described with a general action of the form:

$$S = \int dt \int d^d\boldsymbol{r} \; \mathcal{L}[\phi_n, \partial_\mu \phi_n], \tag{2}$$

where $\mu = t, x, y, z, ..$ and $\mathcal{L}[\phi_n, \partial_\mu \phi_n]$ is the Lagrangian density. By minimizing the action, $\delta S = 0$, one obtains the Euler-Lagrange equations of motion:

$$\sum_\mu \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}[\phi_n, \partial_\mu \phi_n]}{\partial(\partial_\mu \phi_n)} - \frac{\partial \mathcal{L}[\phi_n, \partial_\mu \phi_n]}{\partial \phi_n} = 0 \qquad \forall n. \tag{3}$$

(2.a) Consider the classical ϕ^4 theory of a *real, scalar, non-relativistic* field $\phi \in \mathbb{R}$ described by the action

$$\mathcal{L}[\phi, \partial_\mu \phi] = \frac{1}{2} \sum_\mu (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \alpha \phi^4, \tag{4}$$

and derive the corresponding equations of motion for $\phi(\boldsymbol{r}, t)$.

(2.b) Consider the classical ϕ^4 theory of a *complex, scalar, non-relativistic* field $\phi \in \mathbb{C}$ described by the action

$$\mathcal{L}[\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*] = \frac{1}{2} \sum_\mu |\partial_\mu \phi|^2 - \frac{m^2}{2} |\phi|^2 - \alpha |\phi|^4, \tag{5}$$

and derive the corresponding equations of motion for $\phi(\boldsymbol{r}, t)$ and $\phi^*(\boldsymbol{r}, t)$. You may treat ϕ and ϕ^* as two independent components!

Problem 3 The Gross-Pitaevskii equation

In this problem we study weakly interacting bosons with point-like interactions of strength g in an external trapping potential $V(\boldsymbol{x})$. With m and μ we denote the boson mass and chemical potential, respectively. This system is described by the Hamiltonian ($\hbar = 1$):

$$\hat{\mathcal{H}} = \int d^3\boldsymbol{r} \left\{ \frac{1}{2m} |\boldsymbol{\nabla} \hat{\psi}(\boldsymbol{r})|^2 + (V(\boldsymbol{r}) - \mu)\, \hat{\psi}^\dagger(\boldsymbol{r}) \hat{\psi}(\boldsymbol{r}) \right\} + \frac{g}{2} \int d^3\boldsymbol{r} d^3\boldsymbol{r}'\, \hat{\psi}^\dagger(\boldsymbol{r}) \hat{\psi}^\dagger(\boldsymbol{r}') \delta(\boldsymbol{r} - \boldsymbol{r}') \hat{\psi}(\boldsymbol{r}') \hat{\psi}(\boldsymbol{r}) \tag{6}$$

(3.a) Using the canonical commutation relations of the bosonic field $\hat{\psi}(\boldsymbol{r})$, derive the equations of motion of the field operators $\hat{\psi}(\boldsymbol{r}, t)$ in the Heisenberg picture:

$$i \frac{\partial}{\partial t} \hat{\psi}(\boldsymbol{r}, t) = \left[\hat{\psi}(\boldsymbol{r}, t), \hat{\mathcal{H}} \right] = ... \tag{7}$$

You obtain the operator-valued Gross-Pitaevskii equation.

(3.b) To derive a simpler $\mathbb{C}$-valued classical equation providing an approximate description of the interacting Bose-gas, consider the following variational wavefunction:

$$|\psi(t)\rangle = \frac{1}{\sqrt{N!}} \left(\int d^3\boldsymbol{r}\ \Psi(\boldsymbol{r}, t) \hat{\psi}^\dagger(\boldsymbol{r}) \right)^N |0\rangle. \tag{8}$$

Here N is the total boson number and $\Psi(\boldsymbol{r}, t)$ is a variational parameter depending on space and time. Describe the physical meaning of this state!

(3.c) For the variational ansatz in (3.b), derive the following expectation value,

$$\mathcal{L}[\partial_t \Psi, \boldsymbol{\nabla} \Psi, \Psi] = \langle \psi(t) |\ - i\partial_t + \hat{\mathcal{H}}\ | \psi(t) \rangle \tag{9}$$

which takes the role of a classical Lagrangian density.

(3.d) Derive the Euler-Lagrange equations for $\Psi(\boldsymbol{r}, t)$ from the Lagrangian density $\mathcal{L}$ derived in (3.c). Show that the obtained equation takes the form:

$$i\partial_t \Psi(\boldsymbol{r}, t) = -\frac{1}{2m} \boldsymbol{\nabla}^2 \Psi(\boldsymbol{r}, t) + (V(\boldsymbol{r}) - \mu)\, \Psi(\boldsymbol{r}, t) + g |\Psi(\boldsymbol{r}, t)|^2 \Psi(\boldsymbol{r}, t). \tag{10}$$

Compare this *Gross-Pitaevskii equation* to the operator-valued equation of motion obtained in (3.a)!

Hint: You may treat space as discretize to conceptually simplify your calculations.