Problem 1

- $\Phi \to U\Phi$

  $\Phi^\dagger \Phi \to \Phi^\dagger U^\dagger U \Phi = \Phi^\dagger \Phi$ invariant

  $\to$ no other invariants.

- $\Sigma \to U \Sigma U^\dagger$, $\Sigma = \Sigma^\dagger$, $\mathrm{Tr}\, \Sigma = 0$

  $\mathrm{Tr}\, \Sigma^2$, $\mathrm{Tr}\, \Sigma^3$, $\mathrm{Tr}\, \Sigma^4$ ....

SU(2) case: $rk(SU(2)) = 1 \Rightarrow 1$ invariant

$$\begin{cases} \mathrm{Tr}\, \Sigma^2 \text{ is only invariant} \\ \mathrm{Tr}\, \Sigma^4 \propto (\mathrm{Tr}\, \Sigma^2)^2 \\ \mathrm{Tr}\, \Sigma^3 = 0 \end{cases}$$

$\hookrightarrow$ Check with Pauli

SU(3) case: $rk(SU(3)) = 2 \to 2$ invariants at most

Adjoint $\to$ Gell-Mann Matrices

$\mathrm{Tr}\, \Sigma^2 \to$ invariant

$\mathrm{Tr}\, \Sigma^3 \to$ invariant

$\mathrm{Tr}\,\Sigma^4 \propto (\mathrm{Tr}\,\Sigma^2)^2 \to$ can check explicitly

What about $\det \Sigma$ in both $SU(2)$ & $SU(3)$?

$$\det \Sigma \to \det U^{+}\Sigma U = \det \Sigma$$

Is it new invariant? You can check explicitly that it is not (dimensionality arguments give $\det \Sigma_{SU(2)} \propto \mathrm{Tr}\,\Sigma^2$, $\det \Sigma_{SU(3)} \propto \mathrm{Tr}\,\Sigma^3$ which turns out to be true)

- Φ & Σ

$\to$ $\mathrm{Tr}\,\Sigma^2\,\Phi^{+}\Phi$ & $\Phi^{+}\Sigma\Phi$ & $\Phi^{+}\Sigma^2\Phi$

$$\Phi^{+}\Sigma\Phi \to \Phi^{+}U^{+}U\Sigma U^{+}U\Phi = \Phi^{+}\Sigma\Phi \quad \checkmark$$

$$\Phi^{+}\Sigma^2\Phi \to \Phi^{+}U^{+}U\Sigma U^{+}U\Sigma U^{+}U\Phi = \Phi^{+}\Sigma^2\Phi \quad \checkmark$$

(*) Higher dimensional invariants are not included due to renormalisability requirement

(*) $\Sigma \to -\Sigma \Rightarrow \mathbb{Z}_2$ symmetry forbids $\mathrm{Tr}\,\Sigma^3$ & $\Phi^{+}\Sigma\Phi$

Problem 2

$$V = -\frac{\mu^2}{2} \Phi^T \Phi + \frac{\lambda}{4}(\Phi^T\Phi)^2 \quad , \quad \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

$$= -\frac{\mu^2}{2} \varphi^i\varphi^i + \frac{\lambda}{4}(\varphi^i\varphi^i)^2$$

- $$0 \overset{!}{=} \frac{\partial V}{\partial \varphi^i} = -\mu^2 \varphi^i + \lambda(\varphi^j\varphi^j)\varphi^i = \varphi^i(-\mu^2 + \lambda(\varphi^j\varphi^j))$$

local maximum : $\varphi^i = 0$

global minimum : $(\varphi^i\varphi^i) = v^2 \qquad v = \frac{\mu}{\sqrt{\lambda}}$

↳ vacuum manifold is 2-dimensional sphere

⊛ Choose any point of sphere as ground state

$\varphi_1 = 0 \,,\, \varphi_2 = 0 \,,\, \varphi_3 = v$

→ After SSB, residual rotation in first 2 components

$SO(2)$ i.e.

$\varphi^1 \to \varphi^1 \cos\alpha - \varphi^2 \sin\alpha$

$\varphi^2 \to \varphi^1 \sin\alpha - \varphi^1 \cos\alpha$

$\varphi^3 \to \varphi^3$

SSB = Lagrangian obeys symmetry, vacuum does not

Goldstone's Th: For every spontaneously broken continuous symmetry, theory must contain a massless particle → Nambu Goldstone

To evaluate mass spectrum we evaluate curvature at global minimum

$$\left.\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right|_{\Phi=\begin{pmatrix}0\\0\\v\end{pmatrix}} \rightsquigarrow 3 \text{ by } 3 \text{ Matrix}$$

$$\frac{\partial V}{\partial \varphi^i} = \varphi^i\left(-\mu^2 + \lambda(\varphi^j\varphi^j)\right)$$

$$\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^k} = \delta^{ik}\left(-\mu^2 + \lambda(\varphi^j\varphi^j)\right) + 2\varphi^i \lambda \varphi^k$$

$$\left.\frac{\partial^2 V}{\partial \varphi^i \partial \varphi^k}\right|_{\Phi=\begin{pmatrix}0\\0\\v\end{pmatrix}} = \delta^{ik}\left(-\mu^2 + \lambda v^2\right) + 2\delta^{i3}\delta^{k3}\lambda v^2$$

$$\overset{v^2 = \frac{\mu^2}{\lambda}}{=} 2\delta^{i3}\delta^{k3}\mu^2\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\lambda v^2 \end{pmatrix}$$

→ 1 eigenvalue $2\lambda v^2$, 2 vanishing eigenvalue

$$\Phi = \begin{pmatrix} G_1 \\ G_2 \\ v+h \end{pmatrix} \Rightarrow m_h^2 = 2\lambda v^2, \quad m_{G_1} = m_{G_2} = 0$$

- Generators: $(T_i)_{jk} = -i\,\varepsilon_{ijk}$

$$T_3 \Phi = -i\,\varepsilon_{3jk}\,\Phi_k = -i\,\varepsilon_{3jk}\,\delta_{3k}\,v$$

$$\propto \varepsilon_{3j3} = 0$$

$$T_1 \Phi = -i\,\varepsilon_{1j3}\,v \neq 0$$

Similar $T_2 \Phi \neq 0$

- Gauge Theory:

$$\partial_\mu \Phi \to D_\mu \Phi$$

$$D_\mu = \partial_\mu - ig A_\mu = \partial_\mu + g A_\mu^a T^a$$

$$\mathcal{L} = \frac{1}{2}(D_\mu \Phi)^T (D^\mu \Phi) - V(\Phi) - \frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a$$

→ Write down each term explicitly

$$\mathcal{L} \supset \frac{g^2}{2} (A_\mu^a T^a \Phi)^T (A^{\mu b} T^b \Phi)$$

↳ evaluate @ $\Phi = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$ to read of gauge mass

$$\frac{g^2}{2}\left(A_\mu^a T^a \Phi\right)^T\left(A^{\mu b} T^b \Phi\right) =$$

$$= \frac{g^2}{2} A_\mu^a A^{\mu b} \, \varepsilon^{aj3} v \, \varepsilon^{bj3} v = \frac{g^2}{2} v^2 \left[(A_\mu^1)^2 + (A_\mu^2)^2 \right]$$

$$= \frac{1}{2} M_{ab}^2 A_\mu^a A^{\mu b}$$

$$\Rightarrow \quad m_{A_1} = m_{A_2} = g v$$

$$m_{A_3} = 0$$

as expected The NG. bosons are eaten by longitudinal dof.

Part II

$$V = -\frac{\mu_i^2}{2}\Phi_i^T\Phi_i + \frac{\lambda_i}{4}(\Phi_i^T\Phi_i)^2 + \frac{\lambda_3}{2}\Phi_1^T\Phi_1\Phi_2^T\Phi_2 + \frac{\lambda_4}{2}(\Phi_1^T\Phi_2)^2$$

Choose $\Phi_1^0 = \begin{pmatrix} 0 \\ 0 \\ v_1 \end{pmatrix}$, $\Phi_2^0 = v_2 \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$ → choice based on residual symm.

i.e. $SO(3) \xrightarrow{\Phi_1^0} SO(2) \xrightarrow{\Phi_2^0} ?$

only λ_4 term is sensitive to θ in potential

$$V^0 = f(v_1, v_2) + \lambda_4 v_1^2 v_2^2 \cos^2\theta$$

→ 2 cases $\lambda_4 > 0$ & $\lambda_4 < 0$

a) $\lambda_4 > 0$ → minimum @ $\cos\theta = 0$

$\Rightarrow \Phi_2^0 = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix}$ $\quad SO(3) \xrightarrow[\Phi_0^1]{} SO(2) \xrightarrow[\Phi_0^2]{} 1$

b) $\lambda_4 < 0$ → minimum @ $\sin\theta = 0$

$\Rightarrow \Phi_2^0 = \begin{pmatrix} 0 \\ 0 \\ v_2 \end{pmatrix}$ $\Rightarrow SO(3) \xrightarrow{\Phi_1^0, \Phi_2^0} SO(2)$

2) a) Symmetry breaking pattern => 3 NG bosons

$SO(3) \to 1$

$$\Phi_1 = \begin{pmatrix} G_1 \\ G_2 \\ v_1 + h_1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} G_3 \\ v_2 + h_2 \\ G_4 \end{pmatrix}$$

& expand potential up to quadratic order in the fields. Mass matrix is:

$$\begin{array}{c|cccc} & G_1 & G_2 & G_3 & G_4 \\ \hline G_1 & 0 & 0 & 0 & 0 \\ G_2 & 0 & \lambda_4 v_2^2 & 0 & \lambda_4 v_1 v_2 \\ G_3 & 0 & 0 & 0 & 0 \\ G_4 & 0 & \lambda_4 v_1 v_2 & 0 & \lambda_4 v_1^2 \end{array} \quad (*) \quad \left(\mathcal{L} \supset \frac{1}{2} M^2_{ab} G^a G^b \right)$$

Where extremum conditions are $\mu_1^2 = \lambda_1 v_1^2 + \lambda_3 v_2^2$

$\mu_2^2 = \lambda_2 v_2^2 + \lambda_3 v_1^2$

(*) has 3 zero eigenvalues while a non zero 1

$\to \lambda_4 (v_1^2 + v_2^2)$

The associated eigenvector (diagonalizata) are

$G_1, G_3, \frac{v_2 G_4 - v_1 G_2}{\sqrt{v_1^2+v_2^2}}$ ⇝ massless

&

$\frac{v_1 G_4 + v_2 G_2}{\sqrt{v_1^2+v_2^2}}$ massive $m = \lambda_4(v_1^2+v_2^2)$

For h_i's mass matrix reads

$$\begin{array}{c c} & \begin{array}{cc} h_1 & h_2 \end{array} \\ \begin{array}{c} h_1 \\ h_2 \end{array} & \begin{pmatrix} \lambda_1 v_1^2 & \lambda_3 v_1 v_2 \\ \lambda_3 v_1 v_2 & \lambda_2 v_2^2 \end{pmatrix} \end{array}$$

$\to \mathrm{Tr}(\ ) > 0 \Rightarrow \lambda_1, \lambda_2 > 0$

$\det(\ ) > 0 \Rightarrow \lambda_1 \cdot \lambda_2 > \lambda_3^2$

Spectrum 3 massive & 3 massless

b) Symmetry pattern ⇒ 2N G bosons

$$\Phi_1 = \begin{pmatrix} G_1 \\ G_2 \\ v_1 + h_1 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} G_3 \\ G_4 \\ v_2 + h_2 \end{pmatrix}$$

Mass matrix

$$\begin{array}{c|cccc} & G_1 & G_2 & G_3 & G_4 \\ G_1 & -\lambda_4 v_2^2 & 0 & \lambda_4 v_1 v_2 & 0 \\ G_2 & 0 & -\lambda_4 v_2^2 & 0 & \lambda_4 v_1 v_2 \\ G_3 & \lambda_4 v_1 v_2 & 0 & -\lambda_4 v_1^2 & 0 \\ G_4 & 0 & \lambda_4 v_1 v_2 & 0 & -\lambda_4 v_1^2 \end{array}$$

Extremum condition

$$\mu_1^2 = \lambda_1 v_1^2 + (\lambda_3 + \lambda_4) v_2^2$$

$$\mu_2^2 = \lambda_2 v_2^2 + (\lambda_3 + \lambda_4) v_1^2$$

→ Mass matrix is of rk 2 ⇒ 2 NG boson

Normal modes are:

$$\frac{v_1 G_2 + v_2 G_4}{\sqrt{v_1^2 + v_2^2}} \quad , \; m = 0$$

$$\frac{v_1 G_1 + v_2 G_3}{\sqrt{v_1^2 + v_2^2}} \quad , \; m = 0$$

$$\frac{v_1 G_4 - v_2 G_2}{\sqrt{v_1^2 + v_2^2}} \quad m = -\frac{\lambda_4}{2}(v_1^2 + v_2^2)$$

$$\frac{v_1 G_3 - v_2 G_1}{\sqrt{v_1^2+v_2^2}} \quad , \quad m = -\frac{d_4}{2}(v_1^2+v_2^2)$$

$(d_4 < 0)$

For h_i's, we find

$$\begin{array}{c|cc} & h_1 & h_2 \\ h_1 & d_1 v_1^2 & (d_3+d_4) v_1 v_2 \\ h_2 & (d_3+d_4) v_1 v_2 & d_2 v_2^2 \end{array}$$

meaning $d_1, d_2 > 0$, $d_1 d_2 > (d_3+d_4)^2$

Spectrum = 2 massless & 4 massive.

Gauge spectrum:

$$\Phi_1^0 = \begin{pmatrix} 0 \\ 0 \\ v_1 \end{pmatrix} \quad \Phi_2^0 = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix}$$

a) $\frac{1}{2}(D_\mu \Phi_1^0)^T (D^\mu \Phi_1^0) + \frac{1}{2}(D_\mu \Phi_2^0)^T (D^\mu \Phi_2^0)$

$$\supset \frac{g^2}{2}\left(A_1^2(v_1^2+v_2^2) + A_2^2 v_1^2 + A_3^2 v_2^2\right)$$

3 massive gauge boson (we had 3 NG

as expected)

$$m_{A_1} = g\sqrt{v_1^2+v_2^2} \qquad m_{A_2} = g\, v_1 \,, \quad m_{A_3} = g\, v_2$$

b) $\Phi_1^0 = \begin{pmatrix} 0 \\ 0 \\ v_1 \end{pmatrix}$, $\Phi_2^0 = \begin{pmatrix} 0 \\ 0 \\ v_2 \end{pmatrix}$

$$\mathcal{L} \supset \frac{g^2}{2}\left(A_1^2\left(v_1^2+v_2^2\right) + A_2^2\left(v_1^2+v_2^2\right)\right)$$

$$\Rightarrow m_{A_1} = m_{A_2} = g\sqrt{v_1^2+v_2^2}$$

A_3 massless

Problem 3

$$\mathcal{L} = \partial_\mu \varphi_i^* \partial^\mu \varphi_i + \mu^2 \varphi_i^* \varphi_i - \lambda(\varphi_i^* \varphi_i)^2$$

$i = 1, 2, 3$

$\rightarrow \quad \varphi_i = \frac{\varphi_{i.1} + i\varphi_{i.2}}{\sqrt{2}}$ ⊛

1) symmetry group is SO(6)

2) $\frac{\partial V}{\partial \varphi_j} = -\frac{\mu^2}{2} 2 \varphi_j + \frac{4}{4} \varphi_j \lambda (\varphi_i \varphi_i) = 0$ → normalisation ⊛

$\varphi_1^2 + \varphi_2^2$

Rotation $\Rightarrow \quad \varphi_i = v \delta_{i6}$

$\Rightarrow -\mu^2 + \lambda v^2 = 0$

$SO(6) \xrightarrow{\langle\varphi\rangle} SO(5)$ (same as SO(3))

3) $\varphi = \begin{pmatrix} \vec{G} \\ v + h \end{pmatrix} \qquad G = \begin{pmatrix} G_1 \\ \vdots \\ G_5 \end{pmatrix}$

$$m_{G_i} = 0$$

$$m_{e}^{2} = 2\lambda v^{2}$$