LMU "Introduction to Physics of Neutrinos" Course 2023
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Homework 5

Problem 1

Take the "real" bi-doublet

$$\Phi = \begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} \ , \tag{1}$$

where ϕ is the Standard Model Higgs doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \ , \tag{2}$$

and $\tilde{\phi} = i\sigma_2 \phi^*$.

i) Show that

$$\phi^\dagger \phi = \frac{1}{2} \mathrm{Tr} \left(\Phi^\dagger \Phi \right) \ . \tag{3}$$

ii) Find the most general transformation of Φ that keeps the above expression invariant.

iii) Is the term $\det(\Phi)$ invariant under this transformation?

iv) Use the above results to write down the most general potential for Φ.

Problem 2

Take now a "complex" bi-doublet

$$\Phi = \begin{pmatrix} \tilde{\phi}_1 & \phi_2 \end{pmatrix} \ , \tag{4}$$

where ϕ_i, $i = 1, 2$, are two Standard Model Higgs doublets

$$\phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix} \ , \tag{5}$$

and $\tilde{\phi}_i = i\sigma_2 \phi_i^*$.

i) Compute

$$\mathrm{Tr} \left(\Phi^\dagger \Phi \right) \ , \tag{6}$$

and show that it is invariant under the Standard Model $\mathrm{SU(2)_L}$ symmetry.

ii) What is the maximal symmetry of the above?

iii) Write down the most general potential for Φ invariant under this symmetry. Keep terms which are at most quartic in the fields.

iv) Should a term proportional to $\det(\Phi)$ be included?

v) Define $\tilde{\Phi} = i\sigma_2 \Phi^* i\sigma_2$. Show that

$$\tilde{\Phi} = (\tilde{\phi}_2 \quad \phi_1) \ . \tag{7}$$

vi) Compute

$$\Phi^\dagger \Phi + \tilde{\Phi}^\dagger \tilde{\Phi} \ , \tag{8}$$

and prove it is invariant under $\mathrm{SU(2)_L \times SU(2)_R}$.

vii) Compute

$$\mathrm{Tr}\left(\Phi^\dagger \tilde{\Phi}\right) \ , \tag{9}$$

and compare with $\det(\Phi)$.

viii) Write down the independent invariants depending on Φ. How many are there?

Problem 3

Take two real scalar fields ϕ_L and ϕ_R connected by parity, i.e. $\phi_L \leftrightarrow \phi_R$. Assuming no linear terms, the most general potential reads

$$V = -\frac{\mu^2}{2}\left(\phi_L^2 + \phi_R^2\right) + \frac{\lambda}{4}\left(\phi_L^4 + \phi_R^4\right) + \frac{\lambda'}{2}\phi_L^2\phi_R^2 \ , \tag{10}$$

with $\mu, \lambda, \lambda' > 0$.

1. Find all possible extrema of V.
2. Show that for $\lambda > \lambda'$, the minimum corresponds to $\langle\phi_L\rangle = 0,\ \langle\phi_R\rangle \neq 0$, or vice versa. Prove that by computing explicitly the masses of the physical states.
3. Show that for $\lambda' > \lambda$, the minimum is for $\langle\phi_L\rangle = \langle\phi_R\rangle \neq 0$. Again, compute explicitly the masses of the physical states.

Problem 4

Consider two $SU(2)_L$ and $SU(2)_R$ triplets Δ_L and Δ_R with $B - L = 2$. Take the field Δ_R—called Δ hereafter—and study the resulting breaking.

In other words, we have

$$G = SU(2)_R \times U(1)_{B-L} \ , \tag{11}$$

with $(B-L)\Delta = 2\Delta$, and Δ transforming under $SU(2)$ as

$$\Delta \mapsto U \Delta U^+ \ , \tag{12}$$

with

$$U = e^{i\vec{\theta}\cdot\vec{\sigma}/2} \ . \tag{13}$$

Note also that

$$Tr\Delta = 0 \ . \tag{14}$$

Since the $U(1)$ charge of Δ is 2, it cannot be a hermitian field, i.e. $\Delta^+ \neq \Delta$.

1. Show that the most general potential reads

$$V = -\mu^2 Tr\Delta^+\Delta + \lambda_1 (Tr\Delta^+\Delta)^2 + \lambda_2 (Tr\Delta^2)(Tr\Delta^{+2}) \tag{15}$$

Hint: Show that the term $Tr\Delta^+\Delta\Delta^+\Delta$ is not independent.

2. Show that

$$\langle\Delta\rangle = \begin{pmatrix} z & ir \\ ir & -z \end{pmatrix} \;, \quad z \in \mathbb{C} \;, \quad r \in \mathbb{R} \;. \tag{16}$$

To this end, effectuate the following steps:

(a) Write $\Delta = \Delta_1 + i\Delta_2$, with $\Delta_i^+ = \Delta_i, i = 1, 2$.
Use the freedom $\Delta_1 \mapsto U\Delta_1 U^+$ to show that we can write

$$\langle\Delta_1\rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \;, \quad a \in \mathbb{R} \;. \tag{17}$$

(b) Show that $\langle\Delta_1\rangle$ leaves intact the $U(1)$ symmetry generated by

$$U_3 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \;. \tag{18}$$

(c) Using the remaining U_3 freedom show that Δ_2 can be written as

$$\Delta_2 = \begin{pmatrix} b & r \\ r & -b \end{pmatrix} \;, \quad b, r \in \mathbb{R} \;. \tag{19}$$

3. Find under which conditions one can obtain

$$\langle\Delta\rangle = r \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \;. \tag{20}$$

4. Find the $SU(2)$ transformation U that yields

$$U\langle\Delta\rangle U^+ = v \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;. \tag{21}$$

Express v in terms of r.

5. Show that the form of (12) conserves the charge

$$Q = T_3 + \frac{Y}{2} \;. \tag{22}$$

6. With (12) show that the neutral gauge boson Z_R has the form we found in Lecture XII.

7. Prove that

$$M_{Z_R}^2 = 2\frac{M_{W_R}^2}{1-\tan^2\theta_W} \ , \tag{23}$$

where

$$\tan\theta_W = \frac{g'}{g} \ , \tag{24}$$

and

$$\frac{1}{g'^{\,2}} = \frac{1}{g^2} + \frac{1}{\bar{g}^2} \ . \tag{25}$$

8. Complete the pattern of possible symmetry breaking, i.e. find other possible minima. Show that in every case there is a residual $U(1)$ symmetry.