LMU "Introduction to Physics of Neutrinos" Course 2023
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Homework 3

Problem 1

Multi-generation see-saw mechanism. This problems serves to make sure that you understand the properties of Majorana spinors and the seesaw mechanism of neutrino mass.

We introduce a gauge singlet fermion, the so-called RH neutrino ν_R, per generation and write down the most general Yukawa interaction for n generations

$$\mathcal{L}_Y = \bar{\nu}_R \, \tilde{\Phi}^\dagger \, Y_D \, \ell_L \, + \, \nu_R^T \, C \frac{M_N^\dagger}{2} \, \nu_R + h.c. \tag{1}$$

where we use a compact notation for the following vectors in the generation space

$$\nu_R = \begin{pmatrix} \nu_1 \\ \cdot \\ \cdot \\ \nu_n \end{pmatrix}_R , \qquad \ell_L = \begin{pmatrix} \ell_1 \\ \cdot \\ \cdot \\ \ell_n \end{pmatrix}_L , \tag{2}$$

with the usual notation for the leptonic and Higgs doublets

$$\ell_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L , \qquad \tilde{\Phi} = i\sigma_2 \Phi^* \, . \tag{3}$$

Thus, in (1), Y_D and M_N are $n \times n$ matrices in the space of generations.

- Show that M_N is a symmetric matrix. This is a general property of Majorana mass matrices of the same species of particles.
- Show that (1) is invariant under the SM $SU(2) \times U(1)$ gauge symmetry.
- Introducing

$$N_{iL} \equiv C \bar{\nu}_{iR}^T \, , \tag{4}$$

  show that (1) can be written as

$$\mathcal{L}_Y = N_L^T \, C \, \Phi^T \, i\sigma_2 \, Y_D \, \ell_L \, + \, N_L^T \, C \, \frac{M_N}{2} N_L + h.c. \, . \tag{5}$$

- From the usual form for the Higgs doublet in the unitary gauge

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h + v \end{pmatrix} ,$$

  show that

$$\mathcal{L}_{\text{mass}} = N_L^T \, C M_D \nu_L \; + \; N_L^T \, C \, \frac{M_N}{2} N_L + h.c. \; , \tag{6}$$

where

$$M_D = \frac{Y_D \, v}{\sqrt{2}} \; . \tag{7}$$

- Show next

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} \left[\nu_L^T \, M_D^T \, C \, N_L \; + \; N_L^T \, M_D \, C \, \nu_L \; + \; N_L^T \, M_N \, C \, N_L \right] + h.c.. \tag{8}$$

  In other words, one has a mass matrix (up to a factor 1/2) between ν_L and N_L

$$M_{\nu N} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_N \end{pmatrix} \; . \tag{9}$$

- Assume $M_N \gg M_D$ and show that (9) can be put in block-diagonal form as

$$D_{\nu N} = \begin{pmatrix} M_\nu & 0 \\ 0 & M_N \end{pmatrix} \; , \tag{10}$$

  by a unitary transformation

$$U^T \, M_{\nu N} \, U = D_{\nu N} \; ,$$

  where

$$U = \begin{pmatrix} 1 & \Theta^\dagger \\ -\Theta & 1 \end{pmatrix} \; , \tag{11}$$

  and

$$M_\nu = -M_D^T \, M_N^{-1} \, M_D \; , \tag{12}$$

  with $\Theta = M_N^{-1} \, M_D$, and where only terms up to order Θ^2 are considered.

- Next diagonalize the symmetric matrices M_ν and M_N by the unitary matrices V_L and V_R

$$M_\nu = V_L^* m_\nu V_L^\dagger \; , \tag{13}$$

  and similarly for M_N. The convention above corresponds to V_L being the leptonic mixing matrix (the PMNS matrix) in the basis of diagonal charged leptons

$$\frac{g}{\sqrt{2}} \bar{\nu}_L \gamma^\mu V_L e_L W_\mu^+ \; , \tag{14}$$

  where ν and e stand for all the neutrinos and charged leptons respectively.

- Once obtained, the physical states (eigenstates of the mass matrices) can be cast in the 4-dimensional Majorana form. Do that and show that the factor 1/2 in (8) is not physical.

Problem 2

Let us imagine that the neutrino is a Majorana particle, i.e. $\nu \equiv \nu_L + C\,\overline{\nu_L}^T$. This happens naturally in the seesaw mechanism discussed in the class, when there exists a heavy neutral majorana lepton $N \equiv N_L + C\,\overline{N_L}^T$, with a mass m_N. The neutrino gets a mass through the mixing with N which then leads to N having weak interaction

$$\mathcal{L} = \frac{g}{\sqrt{2}}\theta\, W_\mu^+ \bar{N}\gamma^\mu L e + \text{h.c.}, \tag{15}$$

where L is the usual chiral projector $L = (1+\gamma_5)/2$. In the above formula $\theta = m_D/m_N = \sqrt{m_\nu/m_N}$ is the mixing between ν and N, and $m_D = y_D v$ is the Dirac mass term between ν and N. In the usual notation, v is the vacuum expectation value of the Higgs field and y_D is what we call the neutrino Dirac Yukawa coupling defined through the interaction

$$y_D N_L^T C \nu_L (h+v) + h.c.\ . \tag{16}$$

i. Compute the decay rate for $N \to W^+ + e$. Neglect the electron mass.

ii. What is the result in the limit of vanishing W mass? Does the result depend on how you take the limit, i.e. whether the gauge coupling g goes to zero or the vacuum expectation value v? Can you explain what is going on in both cases?

iii. Is there another decay channel? Recall that N is a Majorana particle, or better half-particle and half-antiparticle.

iv. Compute the lifetime of N.