INTRODUCTION TO PHYSICS OF NEUTRINOS

Solutions to Homework 2

1. Proca theory

$$\mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} m_A^2 A^2$$

1. $$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = 0.$$

$$m_A^2 A_\nu + \partial_\mu F_{\mu\nu} = 0.$$

2. act with ∂_ν.

$$\partial_\mu \partial_\nu F_{\mu\nu} + m_A^2 \partial_\nu A_\nu = 0$$

$$m_A^2 \neq 0 \quad \partial_\nu A_\nu = 0.$$

$$m_A = 0.$$

$$\partial_\mu F_{\mu\nu} = 0.$$

$$\partial_\mu \partial_\mu A_\nu - \partial_\nu \partial \cdot A = 0$$

$$\Box A_0 - \partial_0 (\partial_0 A_0 - \partial_j A_j) = 0.$$

$$-\Delta A_0 + \partial_0 \partial_j A_j = 0 .$$

$$A_0 = \frac{1}{\Delta} \partial_0 \partial_j A_j$$

$$\Box A_j - \partial_j \left(\partial_0^2 \frac{1}{\Delta} \partial_k A_k - \partial_k A_k \right) =$$

$$= \Box A_j - \partial_j \Box \frac{1}{\Delta} \partial_k A_k =$$

$$= \Box \left(\delta_{jk} - \partial_j \frac{1}{\Delta} \partial_k \right) A_k .$$

$$A_k = A^T_k + \partial_k \phi$$

$$\Box A^T_k = 0 .$$

$\uparrow$ 2 d.o.f. $\quad \uparrow$ 1

$A_0 \leftarrow$ 1. constraint

d.o.f. $\underline{2 + 1}$

$$\Box A_\mu - \partial_\mu \partial \cdot A + m^2 A_\mu = 0 .$$

$$(\Box + m^2) A_0 - \partial_0 (\partial_0 A_0 - \partial_k A_k) = 0$$

$$(m^2 - \Delta) A_0 + \partial_0 \partial_k A_k = 0$$

$$A_0 = \frac{1}{\Delta - m^2} \partial_0 \partial_k A_k .$$

$$(\Box + m^2) A_j - \partial_j \left(\partial_0^2 \frac{1}{\Delta - m^2} \partial_k A_k - \right.$$

$$\left. - \partial_k A_k \right) = 0$$

$$(\Box + m^2) \left(\delta_{kj} - \frac{1}{\Delta - m^2} \partial_k \partial_j \right) A_k = 0$$

$$A_k = A^T{}_k + \partial_k \phi .$$

↳ not decouple

3 d.o.f.

$A^T \leftarrow \pm 1$; $\phi \leftarrow 0$. spin

3.

$$\mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} m_A^2 A^2 =$$

$$= -\frac{1}{2} \partial_\mu A_\nu F_{\mu\nu} + \frac{1}{2} m_A^2 A^2 \overset{BT}{=}$$

$$= \frac{1}{2} A_\nu \partial_\mu F_{\mu\nu} + \frac{1}{2} m_A^2 A^2 =$$

$$= \frac{1}{2} A_\nu \left[(\Box + m_A^2) \eta_{\mu\nu} - \partial_\mu \partial_\nu \right) A_\mu$$

$$K = (\Box + m_A^2) \eta_{\mu\nu} - \partial_\mu \partial_\nu$$

$$\tilde{K}_{\mu\nu} = (-p^2 + m_A^2) \eta_{\mu\nu} + p_\mu p_\nu$$

$$\tilde{K}_{\mu\nu} G_{\nu\alpha} = i \eta_{\mu\alpha} .$$

$$G_{\nu\alpha} = A \eta_{\nu\alpha} + B p_\nu p_\alpha .$$

$$\tilde{K}_{\mu\nu} G_{\nu\alpha} =$$

$$= (-p^2 + m_A^2) A \eta_{\mu\alpha} + A p_\mu p_\alpha +$$

$$+ B(-p^2 + m_A^2) p_\mu p_\alpha + B p^2 p_\mu p_\alpha .$$

$$A + m_A^2 B = 0 .$$

$$B = -\frac{A}{m_A^2}$$

$$A = \frac{i}{-p^2 + m_A^2}$$

$$G_{\nu\alpha} = \frac{i}{-p^2 + m_A^2} \eta_{\nu\alpha} - \frac{i}{-p^2 + m_A^2} \frac{p_\nu p_\alpha}{m_A^2}$$

$$G_{\nu\alpha} = \frac{i}{-p^2 + m_A^2}\left(\eta_{\nu\alpha} - \frac{p_\nu p_\alpha}{m_A^2}\right)$$

high energy blow up.

4.

$$\mathcal{L} = \frac{1}{2} A_\mu \left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu \right) A_\nu$$

$$\left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu \right) \partial_\nu = 0 \Leftarrow \text{zero mode.}$$

$$\mathcal{L} = \frac{1}{2} A_\mu \left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu \right) A_\nu +$$

$$+ \frac{1}{2\xi} A_\mu \partial_\mu \partial_\nu A_\nu$$

$$K = \Box \eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu$$

$$\tilde{K} = -p^2 \eta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu$$

$$G = A\,\eta_{\mu\nu} + B\,P_\mu P_\nu .$$

$$\left(-p^2\,\eta_{\mu\nu} + \left(1-\frac{1}{\xi}\right)P_\mu P_\nu\right) .$$

$$\left(A\,\eta_{\nu\alpha} + B\,P_\nu P_\alpha\right) = i\,\eta_{\mu\alpha} .$$

$$-p^2 A\,\eta_{\mu\alpha} - p^2 B\,P_\mu P_\alpha +$$

$$+\left(1-\frac{1}{\xi}\right)A\,P_\mu P_\alpha + \left(1-\frac{1}{\xi}\right)B\,p^2 P_\mu P_\alpha =$$

$$= i\,\eta_{\mu\alpha} .$$

$$\left(1-\frac{1}{\xi}\right)A - \frac{1}{\xi}p^2 B = 0 .$$

$$(\xi - 1)\frac{1}{p^2}A = B .$$

$$A = -\frac{i}{p^2} .$$

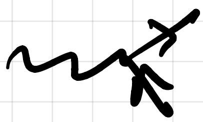
$$G_{\mu\nu} = -\frac{i}{p^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{P_\mu P_\nu}{p^2}\right) .$$

5.

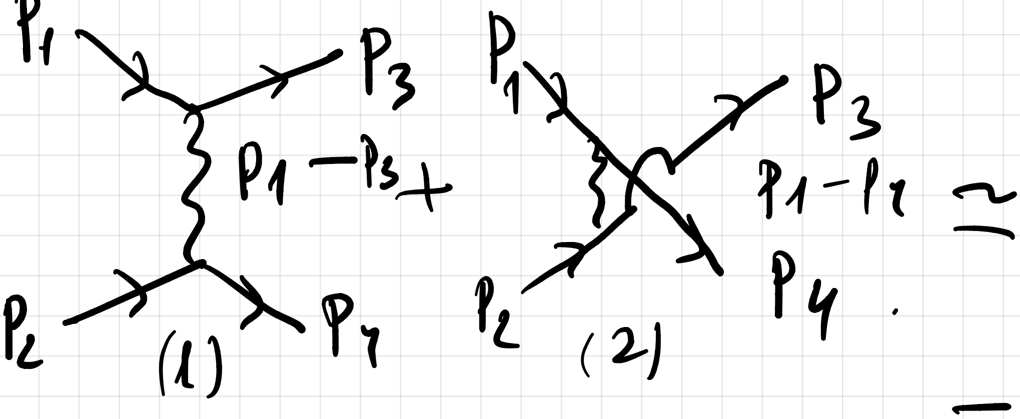
$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F^2 + \frac{1}{2}m_A^2 A^2 +$$
$$+ g\,\bar{\psi}\gamma_\mu L \psi A_\mu .$$

$$\frac{i}{\not{p} - m} .$$

$$\frac{i}{-p^2 + m_A^2}\left(\eta_{\nu\alpha} - \frac{p_\nu p_\alpha}{m_A^2}\right)$$



$$i g \gamma_\mu L$$



$P_1 + P_2 =$
$= P_3 + P_4$
$P_1 - P_3 = -P_2 + P_4$

(1) $$g^2\left(\bar{\psi}(p_3)\gamma_\mu L \psi(p_1)\,\bar{\psi}(p_4)\gamma^\nu L \psi(p_2)\right)$$

$$\times \frac{i}{-(p_1 - p_3)^2 + m_A^2}\left(\eta_{\mu\nu} - \frac{(p_1 - p_3)_\nu (p_1 - p_3)_\mu}{m_A^2}\right) =$$

(2) $$g^2\left(\bar{\psi}(p_4)\gamma_\mu L \psi(p_1)\,\bar{\psi}(p_3)\gamma_\nu L \psi(p_2)\right. \times$$

$$\frac{i}{-(p_1 - p_4)^2 + m_A^2}\left(\eta_{\mu\nu} - \frac{(p_1 - p_4)_\nu (p_1 - p_4)_\mu}{m_A^2}\right)$$

(1) $g^2\, \bar\Psi_3 \gamma_\mu L \Psi_1\, \bar\Psi_4 \gamma_\mu L \Psi_2 \times \frac{i}{-(p_1-p_3)^2 + m_A^2} -$

$L - R =$
$= -\gamma_5.$

$-g^2 \frac{i}{(p_1-p_3)^2 - m_A^2} \frac{1}{m_A^2} \left(m_f \bar\Psi_3 L \Psi_1 - m_f \bar\Psi_3 R \Psi_1 \right) \left(m_f \bar\Psi_4 L \Psi_2 - m_f \bar\Psi_4 R \Psi_2 \right) =$

$= -g^2\, \bar\Psi_3 \gamma_\mu L \Psi_1\, \bar\Psi_4 \gamma_\mu L \Psi_2 \times \frac{i}{(p_1-p_3)^2 - m_A^2} -$

$-g^2 \frac{i}{(p_1-p_3)^2 - m_A^2} \left(\frac{m_f}{m_A}\right)^2 \bar\Psi_3 \gamma_5 \Psi_1\, \bar\Psi_4 \gamma_5 \Psi_2.$

(2) same <u>2 parts</u>.

$m_f \to 0.$

$U(1)$ gauge theory R decouples.

$m_A \to 0 \leftarrow$ not smooth!

We have the Stückelberg Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_A^2 \tilde{A}_\mu^2 ,$$

with

$$\tilde{A}_\mu \equiv A_\mu - \frac{\partial_\mu G}{m_A}$$

① i) For gauge invariance to be restored, we need, in addition to $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$, to also require that

$$G \to G' = G + m_A \alpha .$$

ii) We now introduce the gauge-fixing term in the Lagrangian

$$S' = \int d^4x \, (\mathcal{L} + \mathcal{L}_{g.f.})$$

$$= \int d^4x \Big[-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_A^2 \Big(A_\mu^2 + \frac{1}{m_A^2} (\partial_\mu G)^2 - \frac{2}{m_A} A_\mu \partial_\mu G \Big) - \frac{1}{2\xi} \Big((\partial \cdot A)^2 + \xi^2 m_A^2 G^2 + 2 \xi m_A (\partial \cdot A) G \Big) \Big]$$

$$= \int d^4x \Big[-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_A^2 A_\mu^2 - \frac{1}{2\xi} (\partial \cdot A)^2 + \frac{1}{2m_A^2} (\partial_\mu G)^2 - \frac{\xi m_A^2}{2} G^2 \Big]$$

Varying the above w.r.t. the fields we get their eom:

$$\partial_\mu F_{\mu\nu} + m_A^2 A_\nu - \frac{1}{\xi} \partial_\nu (\partial \cdot A) = 0,$$

$$(\partial^2 + \xi m_A^2) G = 0 .$$

From the above we can immediately find the propagators for A, G:

$$D^{\xi}_{\mu\nu}(k) = \frac{-i}{k^2 - m_A^2}\left(\eta_{\mu\nu} - \frac{1-\xi}{k^2 - \xi m_A^2} k_\mu k_\nu\right)$$

$$D^{\xi}(k) = \frac{i}{k^2 - \xi m_A^2}$$

② i) At high energies, $k \to \infty$, we have

$$D^{\xi}_{\mu\nu}(k) \sim 0 \quad , \quad D^{\xi}(k) \sim 0$$

ii) For $\xi \to \infty$, we obtain

$$D^{\xi\to\infty}_{\mu\nu}(k) = -\frac{i}{k^2 - m_A^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m_A^2}\right)$$

$$D^{\xi\to\infty}(k) = 0 \qquad \text{(decouples)}.$$

$\to$ Proca propagator

③ It's clear that a physical d.o.f. cannot depend on the choice of gauge; however $m_G \propto \sqrt{\xi}$, meaning that it's gauge dependent, so it cannot be physical field.

④ We need only show that $\mathcal{L}_{G,int}$ is gauge-invariant

$$\mathcal{L}'_{G,int} = m_f \bar{\Psi}'_L \Psi'_R e^{iG'/m_A} + h.c.$$

$$= m_f \bar{\Psi}_L e^{-ig\alpha} \Psi_R e^{\frac{ig}{m_A}(G + m_A \alpha)} + h.c$$

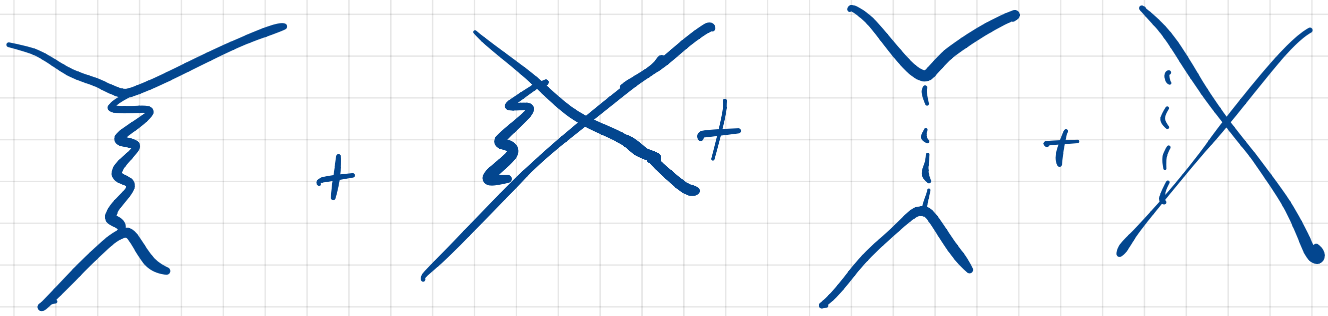
$$= m_f \bar{\Psi}_L \Psi_R e^{\frac{igG}{m_A}} + h.c = \mathcal{L}_{G,int}.$$

(What about the g.f. term? Can you think why $\mathcal{L}'_{gf} = \mathcal{L}_{gf}$?)

$$\mathcal{L}_{G,int} \simeq m_f \bar{\Psi}_L \Psi_R \left(1 + \frac{i g G}{m_A}\right) + h.c.$$

$$\supset i g \frac{m_f}{m_A} \bar{\Psi}_L \Psi_R G + h.c.$$

5. For the $\psi - \psi$ scattering we have the following diagrams:



It's not difficult to see that the results obviously doesn't depend on ξ (that's what G does, kills the ξ-dependence).

6. For $g \to 0$, it's clear that the amplitude for $\psi\psi$ scattering vanishes.

⑦

$$\mathcal{L}_Y = y_f \bar{\Psi}_L \Psi_R \phi + h.c$$

$$= y_f \bar{\Psi}_L \Psi_R (v + h + iG) + h.c.$$

$$= y_f v \bar{\Psi}_L \Psi_R + y_f \bar{\Psi}_L \Psi_R h$$
$$+ i y_f \bar{\Psi}_L \Psi_R G + h.c.$$

$\rightarrow m_f = y_f v$. (1)

At the same time, $m_A = gv$

$\rightarrow v = \frac{m_A}{g}$ (2)

$\rightarrow m_f = y_f \frac{m_A}{g} \rightarrow \boxed{y_f = g \frac{m_f}{m_A}}$

⑧

It's clear that for $g \rightarrow 0$ in the Abelian Higgs model, the

amplitude cannot vanish!
Actually, it receives contributions
from h & G.

⑨ For $v=0$, we have no SSB.

This means that we'll only
have the following interaction:

$$\mathcal{L}_f = y_f \bar{\Psi}_L \Psi_R \phi + h.c.$$

$$\mathcal{L}_2 = g \bar{\Psi}_L \gamma_\mu \Psi_L A_\mu$$

Consistency of the theory
requires that the current is
conserved.

It's also clear that the $\psi-\psi$
process takes place via the
exchange of A_μ & ϕ.