Ludwig-Maximilians-Universität München

SOLUTIONS TO

QCD and Standard Model

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Guidelines:

- The exam consists of 7 problems. Out of Problems 1-6 you need to solve five, the remaining one is a bonus. Problem 7 has to be solved.
- The duration of the exam is 72 hours.
- Please write your name or matriculation number on every sheet that you hand in.
- Your answers should be comprehensible and readable.

GOOD LUCK!

Problem 1	20 P
Problem 2	20 P
Problem 3	20 P
Problem 4	20 P
Problem 5	20 P
Problem 6	20 P
Problem 7	10 P

Total	110 P
Bonus	20 P

Problem 1 (20 points)

a) **Solution [16 P]**

$$
\begin{aligned}
\mathcal{L} =& \ \mathrm{tr}[(\partial_\mu X)^\dagger(\partial^\mu X)] + \mathrm{tr}[(\partial_\mu Y)^\dagger(\partial^\mu Y)] - V \\
V =& \ m_X^2\mathrm{tr}[X^\dagger X] + m_Y^2\mathrm{tr}[Y^\dagger Y] + \lambda_1(\mathrm{tr}[X^\dagger X])^2 + \lambda_2(\mathrm{tr}[Y^\dagger Y])^2 + \\
&+ \lambda_3\mathrm{tr}[(X^\dagger X)^2] + \lambda_4\mathrm{tr}[(Y^\dagger Y)^2] + \lambda_5\mathrm{tr}[X^{\dagger 2}X^2] + \lambda_6\mathrm{tr}[Y^{\dagger 2}Y^2] + \\
&+ \lambda_7\mathrm{tr}[X^\dagger X]\mathrm{tr}[Y^\dagger Y] + \lambda_8(\mathrm{tr}[X^\dagger Y^3] + \mathrm{tr}[XY^{\dagger 3}]) + \lambda_9\mathrm{tr}[X^\dagger XY^\dagger Y] + \\
&+ \lambda_{10}\mathrm{tr}[X^\dagger XYY^\dagger] + \lambda_{11}(\mathrm{tr}[X^\dagger Y^\dagger XY] + \mathrm{tr}[X^\dagger YXY^\dagger]) \\
&+ \lambda_{12}\mathrm{tr}[X^\dagger Y^\dagger YX] + \lambda_{13}\mathrm{tr}[X^\dagger YY^\dagger X] + \lambda_{14}\mathrm{tr}[X^2]\mathrm{tr}[X^{\dagger 2}] + \lambda_{15}\mathrm{tr}[Y^2]\mathrm{tr}[Y^{\dagger 2}] + \\
&+ \lambda_{16}\mathrm{tr}[XY]\mathrm{tr}[X^\dagger Y^\dagger] + \lambda_{17}\mathrm{tr}[XY^\dagger]\mathrm{tr}[X^\dagger Y] + \lambda_{18}(\mathrm{tr}[XY^\dagger]\mathrm{tr}[Y^{\dagger 2}] + \mathrm{tr}[X^\dagger Y]\mathrm{tr}[Y^2])
\end{aligned} \tag{1}
$$

b) **Solution [4 P]**

Define

$$
\begin{aligned}
X \equiv X^a T^a \ , \quad X^\dagger \equiv (X^*)^a T^a \\
Y \equiv Y^a T^a \ , \quad Y^\dagger \equiv (Y^*)^a T^a
\end{aligned} \tag{2}
$$

and use

$$
\begin{aligned}
\mathrm{tr}[T^aT^b] &= \frac{1}{2}\delta^{ab} \\
\mathrm{tr}[T^aT^bT^cT^d] &= \frac{1}{4N}\delta^{ab}\delta^{cd} + \frac{1}{8}(d^{abe}d^{cde} - f^{abe}f^{cde} + if^{abe}d^{cde} + if^{cde}d^{abe})
\end{aligned} \tag{3}
$$

to obtain the EOMs for Z^a, $(Z^*)^a$, where $Z = X, Y$, from the Euler-Lagrange equations

$$
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu Z^a)} - \frac{\partial \mathcal{L}}{\partial Z^a} = 0 \tag{4}
$$

Problem 2 (20 points)

Solution [4 P]

Proton decay violates baryon number. The lowest dimension for an operator leading to proton decay is 6, as such an operator must contain at least 4 fermion fields; there are no such dimension 4 or 5 operators.

Solution [16 P]

There are four dimension-six operators:

$$
\begin{aligned}
&(q_L^T C i\sigma_2 q_L)(q_L^T C i\sigma_2 l_L) \\
&(q_L^T C i\sigma_2 q_L)(u_R^T C e_R) \\
&(q_L^T C i\sigma_2 l_L)(u_R^T C d_R) \\
&(u_R^T C e_R)(u_R^T C d_R)
\end{aligned} \tag{5}
$$

Problem 3

- **Solution [13 P]**

  We have seen that a general CKM matrix has

$$\frac{N(N-1)}{2}\ \text{rotations},$$
$$\frac{N^2-3N+2}{2}\ \text{phases}, \tag{6}$$

  with N the number of generations. For $N = 3$, we get 1 phase and 3 rotations. In the case at hand, the mass matrix (after diagonalization) has the following form

$$m = \begin{pmatrix} \mu_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \mu_2 \end{pmatrix}. \tag{7}$$

  Since the upper block is diagonal, it is invariant under $U(2)$ rotations. Hence, one rotation and the phase are not physical. So, the resulting CKM matrix has two rotations only. This result can be easily deducted from the explicit form of V_{CKM} matrix.

- **Solution [7 P]**

  The CKM matrix is connected to mass diagonalization matrices as follows

$$V_{CKM} = U_u^+ U_d. \tag{8}$$

  If for instance d's are massless, then U_d is arbitrary, we can choose $U_d = U_u$ and $V_{CKM} = 1$.

Problem 4 (20 points)

a) **Solution [6 P]**

$$\tilde{H} = \begin{pmatrix} h^{0*} \\ -h^{+*} \end{pmatrix} \quad \Rightarrow \quad \Phi = \begin{pmatrix} h^{0*} & h^+ \\ -h^{+*} & h^0 \end{pmatrix} \tag{9}$$

b) **Solution [8 P]**

$$H^\dagger H = h^{+*}h^+ + h^{0*}h^0 = \frac{1}{2}\text{tr}[\Phi^\dagger\Phi] = \det[\Phi] \tag{10}$$

Therefore

$$V = -\frac{\mu^2}{2}\text{tr}[\Phi^\dagger\Phi] + \frac{\lambda}{4}(\text{tr}[\Phi^\dagger\Phi])^2 = -\mu^2\det[\Phi] + \lambda(\det[\Phi])^2 \tag{11}$$

c) **Solution [6 P]**

$$\Phi \to U_L \Phi U_R^\dagger\ , \quad (U_L, U_R) \in SU(2)_L \times SU(2)_R \cong SO(4) \tag{12}$$

Problem 5

Solution [20 P] We can choose

$$< H >= \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{13}$$

In that case, the electric charge acts on it as

$$Q < H >= \left(\frac{\tau_3}{2} + \frac{Y_H}{2}\right)\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{14}$$

Requiring that the above vanishes, we get $Y_H = 1$, which makes the photon massless. Now let us require that

$$QS = Y_S S = 0, \tag{15}$$

or otherwise the photon gets mass from the second Higgs. We have 2 choices, we should choose $Y_S = 0$, or $g' = 0$. The first decouples the new field, while the second spoils SM.

Problem 6

- **Solution [6 P]**

  $$\bar{q} i \not{D} q = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R, \tag{16}$$

  which has the symmetry $U(N)_L \times U(N)_R$.

- **Solution [6 P]** $tr F_{\mu\nu} F_{\alpha\beta}$ is a tensor, but $\epsilon_{\mu\nu\alpha\beta}$ is an axial tensor. Under the Lorenz transformations (O) it transforms in the following way

  $$\epsilon \to \Delta(O)\epsilon, \tag{17}$$

  where Δ is a determinant. In the case of Lorenz transformations,

  $$\Delta O = \pm 1. \tag{18}$$

  We get -1 for the P and T transformations. Due to CPT-theorem, T transformation is the same as CP transformation, hence the above term breaks P and CP.

- **Solution [8 P]** The total action is the sum of YM and θ actions.

  $$S = S_{YM} + S_\theta. \tag{19}$$

  If we do variation of the first part, we get the standard equations of motion. The variation of second part has the form

  $$\delta S_\theta = 2\theta \int d^4x \epsilon_{\mu\nu\alpha\beta} Tr F_{\mu\nu} \delta F_{\alpha\beta}, \tag{20}$$

  which after integration by parts simplifies to

  $$\delta S_\theta = -4\theta \int d^4x \epsilon_{\mu\nu\alpha\beta} D_\alpha F_{\mu\nu} \delta A_\beta, \tag{21}$$

  since $D_{[\alpha} F_{\mu\nu]} = 0$, the above expression does not contribute to the equations of motions.

Problem 7 (10 points)

1. **Solution [5 P]**
$\frac{1}{M}(\partial_\mu \phi)\bar{\psi}\gamma^\mu \psi$ preserves parity
2. **Solution [5 P]**
$\frac{1}{M}(\partial_\mu \phi)\bar{\psi}\gamma^\mu \gamma^5 \psi$ breaks parity