(1) The constraint puts the field on the unit sphere.

(2) We have N fields, subject to 1 constraint, meaning that there are $N-1$ independent degrees of freedom.

(3) We start from

$$S = \frac{N}{2t}\int d^2x\, \partial_\mu \sigma_a \partial_\mu \sigma_a .$$

$$= \frac{1}{2}\int d^2x\, \partial_\mu\left(\sqrt{\frac{N}{t}}\,\sigma_a\right)\partial_\mu\left(\frac{\sqrt{N}}{t}\,\sigma_a\right)$$

$$= \frac{1}{2}\int d^2x\, \partial_\mu \pi_a \partial_\mu \pi_a , \quad (*)$$

where $\pi_a = \sqrt{\frac{N}{t}}\,\sigma_a$, meaning that $\sigma_a = \sqrt{\frac{t}{N}}\,\pi_a$. Notice that in terms of the π's, the constraint $\sigma_a\sigma_a = 1$ becomes

$$\pi_a \pi_a = \frac{N}{t}. \quad (**)$$

To enforce (**), we introduce a Lagrange multiplier λ such that (*) becomes

$$S_\lambda^t = \frac{1}{2}\int d^2x \left[\partial_\mu \pi_a \partial_\mu \pi_a - \lambda\left(\pi_a \pi_a - \frac{N}{t}\right)\right]$$

(Notice that if we vary the S_λ^t w.r.t. λ, we obtain (**).)

(4) The generating functional reads

$$Z = \int D\lambda D\pi \, e^{-S_\lambda^t}$$

$$= \int D\lambda D\pi \, e^{-\frac{1}{2}\int d^2x \left[\partial_\mu \pi_a \partial_\mu \pi_a - \lambda\left(\pi_a^2 - \frac{N}{t}\right)\right]}$$

Notice that π_a's appear quadratically, so we can perform the Gaussian integral over the fields to obtain

$$Z = \int \mathcal{D}\lambda \, e^{-S_{eff}(\lambda)} ,$$

with

$$S_{eff}(\lambda) = -\log \det(-\partial^2 - \lambda)^{-N/2} + \frac{N}{2t}\int d^2x \, \lambda$$

$$= \frac{N}{2}\left(\operatorname{tr} \log(-\partial^2 - \lambda) + \frac{1}{t}\int d^2x \, \lambda \right)$$

⑤ The eom for λ is found by considering (λ = const. corresponds to the saddle-point approximation, valid for $N \to \infty$)

$$\frac{\delta}{\delta\lambda} S_{eff}(\lambda) = 0 .$$

In momentum space the above becomes

$$\to \frac{1}{t} - \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - \lambda} = 0 .$$

The momentum integral is easily evaluated with a sharp cutoff Λ as

$$I = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - \lambda} = \int \frac{dk}{2\pi} \frac{k}{k^2 - \lambda} = \frac{1}{4\pi} \log(k^2 - \lambda)\Big|_0^\Lambda$$

$\rightarrow I \simeq \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2}$, for $\Lambda^2 >> \lambda = m^2$

$$\rightarrow \frac{1}{t} = \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2} \qquad \circledast$$

⑥ Let's introduce the running coupling constant $t(\mu)$

$$\frac{1}{t(\mu)} = \frac{1}{t} + 4\pi \log \frac{\mu^2}{\Lambda^2},$$

with μ the renormalization scale.

Using $\circledast$, we find

$$\frac{1}{t(\mu)} = \frac{1}{4\pi} \log \frac{\mu^2}{m^2},$$

meaning that

$$\lambda = m^2 = \mu^2 e^{-4\pi/t(\mu)} .$$

Plugging the above result into S_λ, we observe that indeed, this plays the role of a mass term for the π's.