Higgs phenomenon in $SU(2) \times U(1)$

a) Vaccum Manifold:

Lets Minimize $V(H) = \lambda \left(H^\dagger H - \frac{v^2}{2}\right)^2$

Note that $V(H) \geq 0$, thus if $V(H)=0$, then H is at the minimum of V.

$$V(H)=0 \Rightarrow H^\dagger H - \frac{v^2}{2} = 0$$
$$H^\dagger H = \frac{v^2}{2}$$

Then, the constant field configurations $H(x) = H$, such that $H^\dagger H = \frac{v^2}{2}$, minimize the potential. The set of all such configurations, up to gauge transformations, is the Vaccum manifold:

$$\mathcal{M} = \left\{ H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \middle| \, H_i \in \mathbb{C},\ H^\dagger H = \frac{v^2}{2} \right\} \Big/ G.$$

where $G = SU(2) \times U(1)$. (Note: H and H' are equivalent if there is a gauge transformation $g \in G$, such that $H' = gH$.)

Remark: Since we want to minimize the total energy, W_μ^a and B_μ are pure gauge configurations. For simplicity we set them $W_\mu^{a(v)} = B_\mu^{(v)} = 0$.

Lets choose a ground state, namely $H^{(v)} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ (Known as unitary gauge). An unbroken generator, Q, is a Hermitian matrix such that

$$Q H^{(v)} = 0 \quad \left(\text{equivalently } e^{i\theta Q} H^{(v)} = H^{(v)}\right)$$

For our specific choice (unitary gauge), with $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $c = d = 0$, and from Hermiticity $a = 1$, $c = 0$ (setting $\mathrm{Tr}[Q,Q] = 2$) $\Rightarrow$ $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = T^3 + Y$

where $T^3 = \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus, we see there is only one unbroken generator and correspondingly an unbroken subgroup $U(1)_Q$

b) Lets now write the potential around $H^{(v)}$:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$H^\dagger H = \frac{1}{2}(v+h)^2$$

$$\Rightarrow V(H) = \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2$$
$$= \lambda\left(vh + \frac{h^2}{2}\right)^2$$
$$= \underbrace{\lambda v^2 h^2}_{= \frac{m_h^2}{2} h^2} + \lambda v h^3 + \frac{\lambda}{4} h^4$$

$$\Rightarrow \boxed{m_h = \sqrt{2\lambda}\, v}$$

$$\boxed{V(h) = \frac{m_h^2}{2} h^2 + \frac{m_h^2}{2v} h^3 + \frac{m_h^2}{8v^2} h^4}$$

c)
$$D_\mu H = \partial_\mu H + \left[-i\frac{g}{2} W_\mu^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\frac{g}{2} W_\mu^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{ig}{2} W_\mu^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\frac{g'}{2} B_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] H$$

$$\hookrightarrow D_\mu H = \begin{pmatrix} -\frac{ig}{2\sqrt{2}}(W_\mu^1 - iW_\mu^2)(v+h) \\ -\frac{i}{2\sqrt{2}}(g' B_\mu - g W_\mu^3)(v+h) + \frac{1}{\sqrt{2}}\partial_\mu h \end{pmatrix}$$

Lets introduce $W_\mu^\pm$, Z_μ and A_μ:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right)$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g W_\mu^3 - g' B_\mu\right)$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g B_\mu + g' A_\mu^3\right)$$

In terms of $W_\mu^\pm$, Z^μ and A_μ, the covariant derivative is now

$$\hookrightarrow \quad D_\mu H = \begin{pmatrix} -\frac{igv}{2} W_\mu^+ \\ \frac{1}{\sqrt{2}} \partial_\mu h + \frac{i\sqrt{g^2+g'^2}}{2\sqrt{2}} v Z \end{pmatrix} + \begin{pmatrix} -i g W_\mu^+ h \\ i \frac{\sqrt{g^2+g'^2}}{2\sqrt{2}} Z_\mu h \end{pmatrix}$$

and the kinetic term becomes (to the quadratic part)

$$\left[(D_\mu H)^\dagger (D^\mu H)\right]^{(2)} = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{g^2 v^2}{2} W_\mu^+ W^{\mu -} + \frac{1}{2}\left(\frac{(g^2+g'^2)v^2}{4}\right) Z_\mu^2.$$

From the above, we conclude that W_μ^+, W_μ^- and Z acquire masses $m_{W^\pm}$ and m_Z, respectively.

d)

$$\boxed{\begin{aligned} m_h &= \sqrt{2\lambda}\; v \\ m_{W^\pm} &= \frac{g}{2} v \\ m_Z &= \frac{\sqrt{g^2+g'^2}}{2} v \\ m_A &= 0 \end{aligned}}$$

We conclude by summarizing the symmetry breaking pattern:

$SU(2) \times U(1) \longrightarrow U(1)_Q$

(3 generators) W_μ^a (1 generator) B_μ $\longrightarrow$ 1 generator $A_\mu \rightarrow$ It remains massless

+3 would be Nambu-Goldstone Bosons.

$\hookrightarrow$ They end up being eaten by W_μ^+, W_μ^- and Z_μ