Sheet 10:

Hand-out: Friday, July 01, 2022[1]

Problem 1 Landau parameters

In this problem we consider a fluid of fermions described by the Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\boldsymbol{p},\sigma} \varepsilon_{\boldsymbol{p}} \hat{n}_{\boldsymbol{p},\sigma} + \frac{\lambda}{2} \sum_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma',\boldsymbol{q}} V(q) \hat{c}^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma} \hat{c}^\dagger_{\boldsymbol{p}'+\boldsymbol{q},\sigma'} \hat{c}_{\boldsymbol{p}',\sigma'} \hat{c}_{\boldsymbol{p},\sigma}, \tag{1}$$

where $\varepsilon_{\boldsymbol{p}}$ is the energy of the non-interacting Fermi gas and $\lambda \ll 1$ is the perturbative interaction strength; Moreover,

$$V(q) = \int d^3r \; e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} V(r) \tag{2}$$

denotes the Fourier transform of the interaction $V(r)$.

(1.a) Using first-order perturbation theory in λ, derive the Landau interaction parameters $f^{a,s}_{\boldsymbol{p},\boldsymbol{p}'}$ for the Fermi liquid.

(1.b) Consider the following interactions,

$$V_1(r) = \lambda_1 \delta^{(3)}(\boldsymbol{r}), \qquad V_2(r) = -\lambda_2 \nabla^2 \delta^{(3)}(\boldsymbol{r}), \tag{3}$$

and calculate the Landau parameters.

(1.c) When a uniform external field (chemical potential or magnetic field) is applied, the Fermi liquid responds by becoming polarized. In addition to the free-fermion response, interactions can suppress or enhance the latter, and in extreme cases the system becomes unstable. Since the feedback of interactions is determined by the Landau parameters, the instability can be shown to occur at $F^s_l = -1$ (Pomeranchuk instability, density response) and $F^a_l = -1$ (Stoner instability, spin response).

Taking your results from (1.b) literally, sketch the regions of $\lambda_{1,2}$ where the Fermi surface becomes unstable.

Problem 2 Hubbard-Stratonovich decoupling of the Coulomb interaction - part 1

Here we consider electrons in three dimensions with mass m and Coulomb interactions

$$\hat{\mathcal{H}}_{\text{int}} = \frac{1}{2} \int d^3x d^3x' \; \hat{\rho}(\boldsymbol{x}) \hat{\rho}(\boldsymbol{x}') \frac{e^2}{4\pi\varepsilon_0 |\boldsymbol{x} - \boldsymbol{x}'|}. \tag{4}$$

[1] If you would like to present your solution(s), feel free to send them to Felix Palm until Fri, July 08.

The goal is to perform a Hubbard-Stratonovich decoupling and show that the system can be described by the path integral:

$$Z = \int \mathcal{D}[\psi^*, \psi, \phi] \; \exp\left[-\int_0^\beta d\tau \int d^3x \; \left\{ \psi^* \left(\partial_\tau - \frac{1}{2m}\nabla^2 + e\phi - \mu \right) \psi - \frac{\varepsilon_0}{2}(\nabla\phi)^2 \right\} \right] \quad (5)$$

(2.a) Formulate the path integral for Z starting from Eq. (4).

(2.b) Express the Coulomb interaction in Fourier modes by writing

$$\rho(\boldsymbol{x}) = \int \frac{d^3q}{(2\pi)^3} \; \rho_{\boldsymbol{q}} \; e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \quad (6)$$

and calculating $V(q)$.

(2.c) Add the auxiliary white-noise variable $\phi_{\boldsymbol{q}} = i\tilde{\phi}_{\boldsymbol{q}}$ – integrated over the imaginary axis, i.e. $\int_{-\infty}^{\infty} d\tilde{\phi}_{\boldsymbol{q}}$ in the path integral – with the contribution to the action:

$$Z_\phi = \int \mathcal{D}[\phi] \; \exp\left[-\int_0^\beta d\tau \int \frac{d^3q}{(2\pi)^3} \; \left\{ -\frac{1}{2}\varepsilon_0 q^2 \phi_{\boldsymbol{q}} \phi_{-\boldsymbol{q}} \right\} \right] \quad (7)$$

Show that Z_ϕ is convergent.

(2.d) Before we apply the Hubbard-Stratonovich decoupling, consider a general *repulsive* interaction $\mathcal{H}_{\rm int} = \frac{g}{2}\sum_j A_j^2$ with $g > 0$ and show that it can be replaced by $\sum_j \left(\varphi_j A_j - \frac{\varphi_j^2}{2g} \right)$ when adding the Hubbard-Stratonovich white-noise field $q_j = i\varphi_j + igA_j$.

(2.e) Continue from (2.c) and apply the technique from (2.d) to derive the path-integral in Eq. (5).

Problem 3 Hubbard-Stratonovich decoupling of the Coulomb interaction - part 2

Here we consider electrons in three dimensions with mass m and Coulomb interactions as in Problem 2. Our starting point is the path-integral formulation with the Hubbard-Stratonovich field ϕ in Eq. (5).

(3.a) Perform the fermionic Gaussian integrals $\int \mathcal{D}[\psi^*, \psi]$ in Eq. (5) and derive the effective action $S_{\rm eff}[\phi]$.

(3.b) You may now assume that the saddle-point of $S_{\rm eff}[\phi]$ corresponds to $\phi \equiv 0$. To expand $S_{\rm eff}$ up to quadratic order in ϕ, write

$$\hat{G}^{-1} \equiv \hat{G}_0^{-1} + e\phi, \quad (8)$$

with the free electron propagator $\hat{G}_0^{-1} = \partial_\tau + \hat{\boldsymbol{q}}^2/2m - \mu$. As we are expanding around a saddle-point, the term linear in ϕ vanishes. Calculate all terms of order ϕ^2.
Hint: For operators $\hat{A} = \hat{A}_0 + \hat{\varphi}$ one may expand:

$$\mathrm{tr}\log\hat{A} = \mathrm{tr}\log\hat{A}_0 + \mathrm{tr}\left(\hat{A}_0^{-1}\hat{\varphi}\right) - \frac{1}{2}\mathrm{tr}\left(\hat{A}_0^{-1}\hat{\varphi}\hat{A}_0^{-1}\hat{\varphi}\right) + \mathcal{O}(\hat{\varphi}^3) \quad (9)$$

(3.c) Simplify your results in (3.b) and show that the effective action takes the form

$$S_{\rm eff}[\phi] = \sum_{\omega_n} \int \frac{d^3q}{(2\pi)^3} \; \left\{ -\frac{1}{2}\frac{1}{\tilde{V}(q,\omega_n)} \phi_{\boldsymbol{q}} \phi_{-\boldsymbol{q}} \right\} \quad (10)$$

with an effective screened Coulomb interaction $\tilde{V}(q) = V(q)/\varepsilon(\boldsymbol{q}, \omega_n)$. Derive an expression for $\varepsilon(\boldsymbol{q}, \omega_n)$.