Fakultät für Physik im SoSe 2022
TMP - TA3: Condensed Matter Many-Body-Physics and Field Theory I
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https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_22/TMP-TA3/index.html

# Sheet 8:

Hand-out: Friday, June 17, 2022[1]

**Problem 1** Fermionic coherent states

In this problem we show some basic properties of fermionic coherent states. We will denote by $c$ and $c^*$ the Grassman variables corresponding to the set of fermionic operator $\hat{c}$ and $\hat{c}^\dagger$.

(1.a) Show that the overlap of two coherent states is

$$\langle c^*|c\rangle = e^{c^* c}. \tag{1}$$

(1.b) Show the completeness relation,

$$\int dc^* dc\, |c\rangle\langle c^*| e^{-c^* c} = 1. \tag{2}$$

*Hint:* Start from the left hand side and use the explicit representation $|c\rangle = |0\rangle + |1\rangle c$.

(1.c) Show the trace formula:

$$\mathrm{tr}\,\hat{A} = \int dc^* dc\; e^{-c^* c}\langle -c^*|\hat{A}|c\rangle. \tag{3}$$

*Hint:* Show first that: $\delta_{n,m} = \langle n|m\rangle = \int dc^* dc\; e^{-c^* c}\langle -c^*|m\rangle\langle n|c\rangle$, for $n, m = 0, 1$ labeling Fock states.

**Problem 2** Residue integration

In this problem we use residue integration to calculate observables from the Green's function. We consider a free Fermi gas with spin $S$ in continuum,

$$\hat{\mathcal{H}} = \sum_\sigma \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \varepsilon_{\boldsymbol{k}} \hat{\psi}^\dagger_\sigma(\boldsymbol{k})\hat{\psi}_\sigma(\boldsymbol{k}), \qquad \varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 \boldsymbol{k}^2}{2m}, \tag{4}$$

with Fermi wavevector of length $k_F$.

(2.a) Show that the homogeneous density can be written as:

$$\langle\hat{\rho}(\boldsymbol{x})\rangle \equiv \sum_\sigma \langle \hat{\psi}^\dagger_\sigma(\boldsymbol{x})\hat{\psi}_\sigma(\boldsymbol{x})\rangle = -i(2S+1)\mathcal{G}(\boldsymbol{x}, t = 0^-)|_{\boldsymbol{x}=0}. \tag{5}$$

and the homogeneous kinetic energy density as:

$$\langle\hat{T}(\boldsymbol{x})\rangle \equiv -\frac{\hbar^2}{2m}\sum_\sigma \langle \hat{\psi}^\dagger_\sigma(\boldsymbol{x})\boldsymbol{\nabla}^2_{\boldsymbol{x}}\hat{\psi}_\sigma(\boldsymbol{x})\rangle = i(2S+1)\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2_{\boldsymbol{x}}\mathcal{G}(\boldsymbol{x}, 0^-)|_{\boldsymbol{x}=0}. \tag{6}$$

Here $\mathcal{G}(\boldsymbol{x}, t)$ denotes the two-particle Green's function.

[1] If you would like to present your solution(s), feel free to send them to Felix Palm until Fri, June 24.

(2.b) Write out Fourier transforms to show that:

$$\langle \hat{\rho}(\boldsymbol{x}) \rangle = (2S+1) \int \frac{d^d \boldsymbol{k}}{(2\pi)^d} \left[ \int \frac{d\omega}{2\pi i} e^{i\omega\delta} \frac{1}{\omega - \varepsilon_{\boldsymbol{k}} + i\delta \mathrm{sgn}(k - k_F)} \right], \qquad \delta \to 0^+. \tag{7}$$

and find a similar expression for $\langle \hat{T}(\boldsymbol{x}) \rangle$.

(2.c) Perform residue integrals to show that in $d = 3$ dimensions:

$$\langle \hat{\rho}(\boldsymbol{x}) \rangle = (2S+1) \frac{V_F}{(2\pi)^3}, \qquad \langle \hat{T}(\boldsymbol{x}) \rangle = \frac{3}{5} \varepsilon_F \langle \hat{\rho}(\boldsymbol{x}) \rangle. \tag{8}$$

Here $V_F$ is the volume enclosed by the Fermi surface at the Fermi energy $\varepsilon_F$.

**Problem 3** Spin coherent states

In this exercise, we introduce so-called spin-coherent states, which can be used to define path integrals of quantum-spin Hamiltonians. Spin coherent states are defined by rotating the fully polarized state $|S, S\rangle$ – with $\hat{\boldsymbol{S}}^2 |S, S\rangle = S(S+1)|S, S\rangle$ and $\hat{S}^z |S, S\rangle = S|S, S\rangle$ – by angles $\theta$ around the $y$-axis and $\phi$ around the $z$-axis:

$$|\boldsymbol{\Omega}\rangle = e^{i\hat{S}^z \phi} \; e^{i\hat{S}^y \theta} \; e^{i\hat{S}^z \chi} |S, S\rangle. \tag{9}$$

Here $\boldsymbol{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a unit vector, and $\chi$ is a gauge freedom adding an overall phase. Their following properties will be useful:

$$\frac{2S+1}{4\pi} \int d\boldsymbol{\Omega} \; |\boldsymbol{\Omega}\rangle\langle\boldsymbol{\Omega}| = 1, \qquad \mathrm{tr}\hat{A} = \frac{2S+1}{4\pi} \int d\boldsymbol{\Omega} \; \langle \boldsymbol{\Omega} | \hat{A} | \boldsymbol{\Omega} \rangle \tag{10}$$

where $d\boldsymbol{\Omega} = d\theta \sin\theta d\phi$, and:

$$\langle \boldsymbol{\Omega} | \boldsymbol{\Omega}' \rangle = \left( \frac{1 + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}'}{2} \right)^S e^{-iS\psi}, \quad \psi = 2 \arctan \left[ \tan\left( \frac{\phi - \phi'}{2} \right) \frac{\cos[(\theta + \theta')/2]}{\cos[(\theta - \theta')/2]} \right] + \chi - \chi'. \tag{11}$$

(3.a) Construct the spin-coherent path integral, i.e. show that:

$$Z \equiv \mathrm{tr} \mathcal{T}_\tau e^{-\int_0^\beta d\tau \; \hat{\mathcal{H}}(\tau)} = \int \mathcal{D}\boldsymbol{\Omega}(\tau) \; \exp\left( -\mathcal{S}[\boldsymbol{\Omega}(\tau)] \right) \tag{12}$$

with the action:

$$\mathcal{S}[\boldsymbol{\Omega}(\tau)] = -iS \sum_j \omega[\boldsymbol{\Omega}_j] + \int_0^\beta d\tau \; \langle \boldsymbol{\Omega}(\tau) | \hat{\mathcal{H}}(\tau) | \boldsymbol{\Omega}(\tau) \rangle. \tag{13}$$

Here $j$ labels different spins in a lattice, and we defined

$$\exp\left[ iS\omega[\boldsymbol{\Omega}] \right] = \prod_{n=1}^{N} \langle \boldsymbol{\Omega}(\tau_n + \delta\tau) | \boldsymbol{\Omega}(\tau_n) \rangle, \qquad \tau_n = n\delta\tau, \;\; \delta\tau = \beta/N. \tag{14}$$

Which boundary conditions does $\boldsymbol{\Omega}(\tau)$ in the path integral obey?

(3.b) Simplify the contribution $\omega[\mathbf{\Omega}]$ to the action by assuming continuous differentiable trajectories and show that:

$$\omega[\mathbf{\Omega}] = -\int_0^\beta d\tau\ (\partial_\tau \phi)\cos[\theta] + \partial_\tau \chi. \tag{15}$$

By choosing the gauge convention $\chi_j(\tau) \equiv 0$, simplify the result further and show that:

$$\omega[\mathbf{\Omega}] = -\oint_{\phi_0}^{\phi_\beta} d\phi\ \cos(\theta). \tag{16}$$

Discuss how this Berry-phase contribution is geometric and does not depend on the explicit time-dependence of $\phi(\tau)$.

(3.c) Construct the effective action $S$ for a Heisenberg interaction:

$$\hat{\mathcal{H}} = J \sum_{\langle i,j \rangle} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j. \tag{17}$$

*Hint:* Show first that $\mathbf{\Omega} \cdot \hat{\boldsymbol{S}} |\mathbf{\Omega}\rangle = S|\mathbf{\Omega}\rangle$.

(3.d) Discuss on generic grounds and using the above path-integral formalism why $S \to \infty$ corresponds to the classical limit.