LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik im SoSe 2022
TMP - TA3: Condensed Matter Many-Body-Physics and Field Theory I
Dozent: Prof. Dr. Fabian Grusdt
Übungen: Felix Palm, Henning Schlömer

https://www2.physik.uni-muenchen.de/lehre/vorlesungen/sose_22/TMP-TA3/index.html

Sheet 7:

Hand-out: Friday, June 10, 2022[1]

Problem 1 Green's functions

In this problem we calculate some important Green's functions which we saw in the lecture.

(1.a) For a bosonic field $\hat{\phi}_{\boldsymbol{q}} = \sqrt{\hbar/(2m\omega_{\boldsymbol{q}})}\left(\hat{a}_{\boldsymbol{q}} + \hat{a}^\dagger_{-\boldsymbol{q}}\right)$ and a Hamiltonian $\hat{\mathcal{H}}_0 = \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left(\hat{a}^\dagger_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}} + 1/2\right)$, show that

$$D(\boldsymbol{q}, t) \equiv -i\langle 0|\mathcal{T}\hat{\phi}_{\boldsymbol{q}}(t)\hat{\phi}_{-\boldsymbol{q}}(0)|0\rangle = -i\frac{\hbar}{2m\omega_{\boldsymbol{q}}}\left[\theta(t)e^{-i\omega_{\boldsymbol{q}}t} + \theta(-t)e^{i\omega_{\boldsymbol{q}}t}\right], \tag{1}$$

and

$$D(\boldsymbol{q}, \nu) = \frac{\hbar}{2m\omega_{\boldsymbol{q}}}\left[\frac{1}{\nu - (\omega_{\boldsymbol{q}} - i0^+)} + \frac{1}{-\nu - (\omega_{\boldsymbol{q}} - i0^+)}\right]. \tag{2}$$

(1.b) For a fermionic field $\hat{c}_{\boldsymbol{k},\sigma}$ and a Hamiltonian $\hat{\mathcal{H}}_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}}\hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma}$ with ground state $|\psi_0\rangle = \prod_{\sigma,|\mathbf{k}|<k_F}\hat{c}^\dagger_{\mathbf{k},\sigma}|0\rangle$, show that

$$G_{\sigma,\sigma'}(\boldsymbol{k}, \boldsymbol{k}'; t) \equiv -i\langle\psi_0|\mathcal{T}\hat{c}_{\boldsymbol{k},\sigma}(t)\hat{c}^\dagger_{\boldsymbol{k}',\sigma'}(0)|\psi_0\rangle = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\sigma,\sigma'}\begin{cases} -i\theta(|\boldsymbol{k}| - k_F)e^{-i\varepsilon_{\boldsymbol{k}}t} & t > 0 \\ i\theta(k_F - |\boldsymbol{k}|)e^{-i\varepsilon_{\boldsymbol{k}}t} & t < 0 \end{cases} \tag{3}$$

and

$$G(\boldsymbol{k}, \omega) = \frac{1}{\omega - \varepsilon_{\boldsymbol{k}} + i0^+\mathrm{sgn}(\varepsilon_{\boldsymbol{k}})}. \tag{4}$$

Here k_F denotes the Fermi momentum.

Problem 2 Using Grassman integrals

In this exercise, we use Grassman integrals to prove the following identity:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left[A - BD^{-1}C\right]\ \det D, \tag{5}$$

for square matrices A, D of size $N \times N$ and $M \times M$ respectively; B and C are matrices of corresponding sizes. To this end, recall first that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1}^{N} d\alpha_j^* d\alpha_j \prod_{k=1}^{M} d\beta_k^* d\beta_k\ \exp\left[(\alpha^*, \beta^*)\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right], \tag{6}$$

with vectors of Grassman numbers $\alpha, \alpha^*, \beta, \beta^*$ of lengths N, N, M, M, respectively.

[1] If you would like to present your solution(s), feel free to send them to Henning Schlömer until Fri, June 17.

(2.a) Separate the expression Eq (6) into an inner and an outer integral, by writing

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \int \prod_{j=1}^{N} d\alpha_j^* d\alpha_j \; \exp[-\alpha^* A\alpha] \, Y[\alpha^*, \alpha], \tag{7}$$

and find an expression for $Y[\alpha^*, \alpha]$ as a Grassman integral over β^* and β.

(2.b) Solve the inner integral and show that its result is given by

$$Y[\alpha^*, \alpha] = \det(D) \;\; \exp\left[\alpha^* B D^{-1} C\alpha\right]. \tag{8}$$

Hint: Use the following Gaussian Grassman integral:

$$\int \prod_j d\eta_j^* d\eta_j \;\; \exp\left[-\eta^* A\eta + j^*\eta + \eta^* j\right] = \det(A) \;\; \exp[j^* A^{-1} j], \tag{9}$$

for matrix A and vectors of Grassman numbers j and j^*.

(2.c) Use the result from (2.b) to solve the outer integral in (2.a). This way, show the identity Eq. (5).

Problem 3 Generating functionals

In this problem we derive the relation between the free-particle S-matrix and the generating functional containing the Green's function:

$$S[\eta^*, \eta] \equiv \langle 0|\mathcal{T} \;\exp\left[-i\int_{-\infty}^{\infty} dt \;\; \left(\eta^*(t)\hat{\psi}(t) + \hat{\psi}^\dagger(t)\eta(t)\right)\right] |0\rangle =$$
$$= \exp\left[-i\int_{-\infty}^{\infty} dt \; \eta^*(t) G(t-t')\eta(t)\right]. \tag{10}$$

Here, $\hat{\psi}$ is a bosonic or fermionic field operator and $\eta(t)$ is a time-dependent $\mathbb{C}$ or Grassman number, respectively.

(3.a) Start with the case where $\hat{\psi}(t) = \hat{a}(t) = e^{-i\omega t}\hat{a}$ is a bosonic field in the interaction picture as introduced in the lecture. First, introduce N small time steps $\Delta\tau = 2\tau/N$ to write out the time-ordered exponential in the first line of Eq. (10), where the integration limits are from $-\tau$ to τ, and later $N \to \infty$, $\tau \to \infty$.

(3.b) In (3.a) you obtain a product over many exponentials. Factorize each exponential into parts containing only $\hat{a}$ and $\hat{a}^\dagger$ operators, respectively.

Hint: For $\hat{A}$ and $\hat{B}$ with $\left[[\hat{A}, \hat{B}], \hat{A}\right] = \left[[\hat{A}, \hat{B}], \hat{B}\right] = 0$, it holds:

$$\exp\left[\hat{A} + \hat{B}\right] = \exp\left[\hat{B}\right] \exp\left[\hat{A}\right] \exp\left[[\hat{A}, \hat{B}]/2\right] \tag{11}$$

(3.c) Use the result from (3.b) to normal-order the expression. This will allow you to evaluate $S[\eta^*(t), \eta(t)]$ explicitly.

(3.d) Compare your result in (3.c) with the second line of Eq. (10) to show the equality of both expressions in Eq. (10). You may use that for free bosons

$$G(t-t') = -i\theta(t-t')e^{-i\omega(t-t')}. \tag{12}$$

(3.e) Repeat (3.a) - (3.d) for the fermionic driven oscillator with $\hat{\psi}(t) = \hat{c}(t) = e^{-i\epsilon t}\hat{c}$ and Grassman numbers $\eta(t)$!

Hint: Do the calculation separately for $\epsilon > 0$ (particles) and $\epsilon < 0$ (holes), where in the latter case the ground state is not $|0\rangle$, but $|\psi_0\rangle = c^\dagger|0\rangle$ (hole vacuum). Thus, in 3(c) use anti-normal-order for $\epsilon < 0$.