Sheet 6:

Hand-out: Friday, June 03, 2022[1]

Problem 1 Solution of the XY model in a field

In this problem we solve the general XY spin-chain in an external field, described by the Hamiltonian

$$\hat{\mathcal{H}} = -\frac{1}{2}\sum_{j=1}^{L}\Big[(1+\Delta)\,\hat{\sigma}_j^x\hat{\sigma}_{j+1}^x + (1-\Delta)\,\hat{\sigma}_j^y\hat{\sigma}_{j+1}^y + 2B\hat{\sigma}_j^z\Big]. \tag{1}$$

Assume periodic boundary conditions, i.e $\hat{\sigma}^\mu_{L+1} \equiv \hat{\sigma}_1$.

(1.a) Define fermionic operators $\hat{c}_j$ by attaching a Jordan-Wigner string $\hat{F}_j$ to the spin operators. Show that the Jordan-Wigner string can be written as

$$\hat{F}_j = \prod_{i=1}^{j} -1\left(1 - 2\hat{c}_i^\dagger\hat{c}_i\right). \tag{2}$$

(1.b) Show that the Hamiltonian commutes with the parity operator $\hat{P} = \prod_{j=1}^{L}\hat{\sigma}_j^z$,

$$[\hat{\mathcal{H}}, \hat{P}] = 0. \tag{3}$$

Express $\hat{P}$ in terms of the Jordan-Wigner fermions introduced in (1.a).

(1.c) Express the Hamiltonian $\hat{\mathcal{H}}_{\mathrm{OBC}}$ assuming open boundary conditions in terms of the new fermionic operators $\hat{c}_j$, assuming general parameters Δ and B.

(1.d) Because $[\hat{\mathcal{H}}, \hat{P}] = 0$, as shown in (1.b), the Hilbertspace $\mathscr{H}$ can be decomposed into a direct sum of two subspaces $\mathscr{H}_\pm$ of even $(P = +1)$ and odd $(P = -1)$ parity, $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$. Treat these two cases separately and express the spin-spin interactions $\hat{\mathcal{H}}_{\mathrm{B}}$ between sites $j = L$ and $j = 1$, i.e. across the boundary, in terms of Jordan Wigner fermions.

Hint: In one case you obtain periodic ($\hat{c}_{L+1} = \hat{c}_1$), in the other case anti-periodic ($\hat{c}_{L+1} = -\hat{c}_1$) boundary conditions!

(1.e) Show that the Hamiltonian with periodic boundary conditions, $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathrm{OBC}} + \hat{\mathcal{H}}_{\mathrm{B}}$, can be written as:

$$\hat{\mathcal{H}} = \frac{1+P}{2}\hat{\mathcal{H}}_{\mathrm{F}}^{\mathrm{ap}} + \frac{1-P}{2}\hat{\mathcal{H}}_{\mathrm{F}}^{\mathrm{per}}, \tag{4}$$

where $\hat{\mathcal{H}}_{\mathrm{F}}^{\mathrm{per}}$ ($\hat{\mathcal{H}}_{\mathrm{F}}^{\mathrm{ap}}$) denote the fermionic Hamiltonians with periodic (anti-periodic) boundary conditions.

[1] If you would like to present your solution(s), feel free to send them to Felix Palm until Fri, June 10.

(1.f) Diagonalize the fermionic Hamiltonians $\hat{\mathcal{H}}_\mathrm{F}$ by working in Fourier modes and using a Bogoliubov transformation. Show that its spectrum takes the form

$$\omega_k = 2\sqrt{(B+\cos k)^2 + \Delta^2 \sin^2 k}, \tag{5}$$

and derive which discrete momentum values k_n the fermions may occupy if the obey periodic (anti-periodic) boundary conditions, respectively.

Problem 2 The Cooper pair wavefunction

In this problem we derive Cooper's expression for the binding energy of a single Cooper pair. Consider the following Hamiltonian,

$$\hat{\mathcal{H}} = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}}\ \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma} + \hat{\mathcal{H}}_\mathrm{int} \tag{6}$$

as discussed in the lecture.

(2.a) Start from a Fermi-sea $|\mathrm{FS}\rangle$ and make Cooper's ansatz for a state with two more electrons,

$$|\Psi\rangle = \hat{\Lambda}^\dagger|\mathrm{FS}\rangle \qquad \hat{\Lambda}^\dagger = \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}}\ \hat{c}^\dagger_{\boldsymbol{k},\downarrow}\hat{c}^\dagger_{-\boldsymbol{k},\downarrow}. \tag{7}$$

Show that (k_F is the Fermi momentum):

$$|\Psi\rangle = \sum_{|\boldsymbol{k}|>k_F} \phi_{\boldsymbol{k}}\ |\boldsymbol{k}_P\rangle, \qquad \text{with} \quad |\boldsymbol{k}_P\rangle = \hat{c}^\dagger_{\boldsymbol{k},\downarrow}\hat{c}^\dagger_{-\boldsymbol{k},\downarrow}\ |\mathrm{FS}\rangle. \tag{8}$$

In the following exercises we will assume that the Fermi energy $\epsilon_F = \epsilon(k_F) = 0$.

(2.b) Assume that $|\Psi\rangle$ is an eigenstate of $\hat{\mathcal{H}}$, i.e. $\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle$. By comparing components of this vector equation on both sides, show that

$$E\phi_{\boldsymbol{k}} = 2\varepsilon_{\boldsymbol{k}}\ \phi_{\boldsymbol{k}} + \sum_{|\boldsymbol{k}'|>k_F} \langle \boldsymbol{k}_P|\hat{\mathcal{H}}_\mathrm{int}|\boldsymbol{k}'_P\rangle\ \phi_{\boldsymbol{k}'} \tag{9}$$

(2.c) Simplify the interaction by making Cooper's seminal ansatz,

$$V_{\boldsymbol{k},\boldsymbol{k}'} \equiv \langle \boldsymbol{k}_P|\hat{\mathcal{H}}_\mathrm{int}|\boldsymbol{k}'_P\rangle = \begin{cases} -g_0/V & |\varepsilon_{\boldsymbol{k}}|, |\varepsilon_{\boldsymbol{k}'}| < \omega_\mathrm{D} \\ 0 & \text{else} \end{cases} \tag{10}$$

Here ω_D describes a narrow energy shell and $V = L^d$ denotes the system's volume. Using this simplified interaction, show that Eq. (9) becomes:

$$\phi_{\boldsymbol{k}} = -\frac{g_0/V}{E-2\varepsilon_{\boldsymbol{k}}} \sum_{0<\varepsilon_{\boldsymbol{k}'}<\omega_\mathrm{D}} \phi_{\boldsymbol{k}'}. \tag{11}$$

(2.d) From Eq. (11) derive a self-consistency equation for the energy E of the Cooper pair! Take the continuum limit by replacing $\frac{1}{V}\sum_{0<\varepsilon_{\boldsymbol{k}}} \to N(0)\int_0^{\omega_\mathrm{D}} d\varepsilon$, where $N(0)$ is the density of states per spin per unit volume at the Fermi energy, and show that:

$$1 = g_0 N(0) \int_0^{\omega_\mathrm{D}} d\varepsilon\ \frac{1}{2\varepsilon - E} \tag{12}$$

(2.e) Solve Eq. (12) for E, by assuming $2\omega_\mathrm{D} - E \approx 2\omega_\mathrm{D}$. Show that:

$$E = -2\omega_\mathrm{D}\ e^{-\frac{2}{g_0 N(0)}}. \tag{13}$$