Consider translationally invariant MPS, e.g. infinite system, or length-N chain with periodic boundary conditions. Then all tensors defining the MPS are identical: $A_{[\ell]} = A$ for all ℓ.

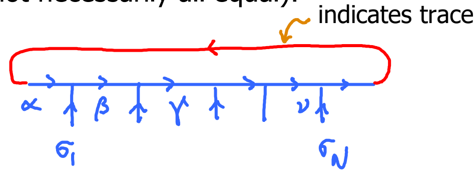
Goal: compute matrix elements and correlation functions for such a system.

1. Transfer matrix

Consider length-N chain with periodic boundary conditions (and A's not necessarily all equal):

$$|\psi\rangle = |\vec{\sigma}_N\rangle \, A_{[1]}^{\alpha\sigma_1}{}_{\beta} \, A_{[2]}^{\beta\sigma_2}{}_{\gamma} \cdots A_{[N]}^{\lambda\sigma_N}{}_{\alpha}$$

$$\equiv |\vec{\sigma}_N\rangle \, \mathrm{Tr}\left[A_{[1]}^{\sigma_1} A_{[2]}^{\sigma_2} \cdots A_{[N]}^{\sigma_N} \right]$$

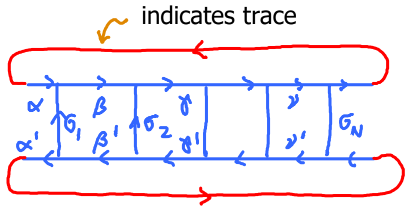


All bonds have same dimension: $D_\alpha = D_\beta = D_\gamma = D_\nu \equiv D$

This is assumed throughout below. (1)

Normalization:

$$\langle\psi|\psi\rangle =$$



(2)

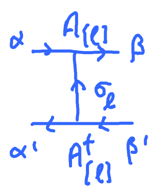
$$= A^{\dagger\alpha'}_{[N]\sigma_N \nu'} \cdots A^{\dagger\gamma'}_{[2]\sigma_2 \beta'} A^{\dagger\beta'}_{[1]\sigma_1\alpha'} A_{[1]}^{\alpha\sigma_1}{}_{\beta} A_{[2]}^{\beta\sigma_2}{}_{\gamma} \cdots A_{[N]}^{\nu\sigma_N}{}_{\alpha} \tag{3}$$

regroup

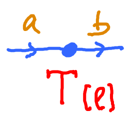
$$= \underbrace{\left(A^{\dagger\beta'}_{[1]\sigma_1\alpha'} A_{[1]}^{\alpha\sigma_1}{}_{\beta}\right)}_{\equiv T_{[1]\alpha'}{}^{\alpha\beta'}{}_{\beta}} \underbrace{\left(A^{\dagger\gamma'}_{[2]\sigma_2\beta'} A_{[2]}^{\beta\sigma_2}{}_{\gamma}\right)}_{\equiv T_{[2]\beta'}{}^{\beta\gamma'}{}_{\gamma}} \cdots \underbrace{\left(A^{\dagger\alpha'}_{[N]\sigma_N\nu'} A_{[N]}^{\nu\sigma_N}{}_{\alpha}\right)}_{\equiv T_{[N]\nu'}{}^{\nu\alpha'}{}_{\alpha}} \tag{4}$$

We defined the 'transfer matrix' (with collective indices chosen to reflect arrows on effective vertex)

$$T_{[\ell]}{}^{a}{}_{b} \equiv T_{[\ell]\underbrace{\alpha'}_{a}}{}^{\underbrace{\alpha\beta'}_{a}}{}_{\underbrace{\beta}_{b}} \equiv A^{\dagger\beta'}_{[\ell]\sigma_\ell\alpha'} A_{[\ell]}^{\alpha\sigma_\ell}{}_{\beta} =$$



$=$



(5)

Note: $D_\mu = D^2$ (6)

Then

$$\langle\psi|\psi\rangle = T_{[1]}{}^{a}{}_{b} \, T_{[2]}{}^{b}{}_{c} \cdots T_{[N]}{}^{n}{}_{a} = \mathrm{Tr}\left(T_{[1]} T_{[2]} \cdots T_{[N]}\right) \tag{7}$$

Assume all A -tensors are identical, then the same is true for all T-matrices. Hence

$$\langle\psi|\psi\rangle = \mathrm{Tr}\left(T^N\right) = \sum_j \left(t_j\right)^N \xrightarrow{N\to\infty} \left(t_1\right)^N \tag{8}$$

where t_j are the eigenvalues of the transfer matrix, and t_1 is the largest one of these.

where t_j are the eigenvalues of the transfer matrix, and t_1 is the largest one of these.

Assume now that A -tensor is left-normalized (analogous discussion holds if it is right-normalized).

Then we know that the MPS is normalized to unity: $1 \overset{\text{(MPS-I.1.22)}}{=} \langle \psi | \psi \rangle$ (1)

(MPS-IV.1.8) implies for largest eigenvalue of transfer matrix: $(t_1)^N = 1 \Rightarrow t_1 = 1.$ (2)

Hence, all eigenvalues of transfer matrix satisfy $\boxed{|t|_j \leq 1}$. (3)

eigenvector label: j = 1
components of eigenvector

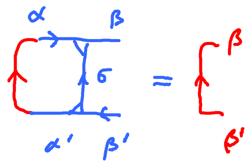
Claim: the left eigenvector with eigenvalue $t_{j=1} = 1$, say $V^{j=1}$, is $(V^1)_a \equiv \mathbb{1}^{\alpha'}{}_{\alpha}$ (4)

Check: do we find $V_a T^a{}_b = V_b$?

'vector in transfer space' = 'matrix in original space'

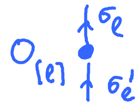
$$V_a T^a{}_b = A^{\dagger \beta'}{}_{\sigma \alpha'} \, \mathbb{1}^{\alpha'}{}_{\alpha} \, A^{\alpha \sigma}{}_{\beta} \qquad (5)$$

$$= A^{\dagger \beta'}{}_{\sigma \alpha} \, A^{\alpha \sigma}{}_{\beta} = \mathbb{1}^{\beta'}{}_{\beta} = V_b \;\checkmark \qquad (6)$$

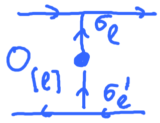


Correlation functions

Consider local operator: $\hat{O}_{[\ell]} = |\sigma_\ell'\rangle \, O_{[\ell]}{}^{\sigma_\ell'}{}_{\sigma_\ell} \, \langle \sigma_\ell|$

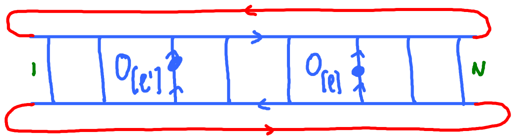


Define corresponding transfer matrix: $T_{O_{[\ell]}} = A^{\dagger}_{\sigma_\ell'} \, O_{[\ell]}{}^{\sigma_\ell'}{}_{\sigma_\ell} \, A^{\sigma_\ell}$



Correlator:

$$C_{\ell'\ell} \equiv \langle \psi | \hat{O}_{[\ell']} \hat{O}_{[\ell]} | \psi \rangle =$$



$$= \mathrm{Tr}\left(T^{\ell'-1} T_{O[\ell']} T^{\ell-\ell'-1} T_{O[\ell]} T^{N-\ell}\right) = \mathrm{Tr}\left(T^{N-(\ell-\ell')-1} T_{O[\ell']} T^{\ell-\ell'-1} T_{O[\ell]}\right)$$

cyclic invariance of trace

Let V^j, t_j be left eigenvectors, eigenvalues of transfer matrix: $V^j T = t_j V^j$

[or explicitly, with matrix indices: $(V^j)_a T^a{}_b = t_j (V^j)_b$]

Transform to eigenbasis of transfer matrix:

$$C_{\ell'\ell} = \sum_{jj'} (t_{j'})^{N-(\ell-\ell')-1} (T_{O[\ell']})^{j'}{}_j \, (t_j)^{\ell-\ell'-1} (T_{O[\ell]})^{j}{}_{j'}$$

For $N \to \infty$, only contribution of largest eigenvalue, $t_{j'} = t_1$, survives from sum over j' :

$$C_{\ell'\ell} \xrightarrow{N\to\infty} t_1^N \sum_j (T_{O[\ell']})^1{}_j \left(\frac{t_j}{t_1}\right)^{\ell-\ell'-1} (T_{O[\ell]})^j{}_1$$

Assume $\hat{O}_{[\ell]} = \hat{O}^\dagger_{[\ell']} \equiv \hat{O}$, and take their separation to be large, $\ell - \ell' \to \infty$

$$C_{\ell'\ell} \xrightarrow{\ell-\ell'\to\infty} t_1^N \left[|(T_O)^1{}_1|^2 + |(T_O)^1{}_2|^2 \left(\frac{t_2}{t_1}\right)^{\ell-\ell'-1} + \dots \right]$$

$$\frac{C_{\ell'\ell}}{\langle\psi|\psi\rangle} = \frac{\langle\psi| O_{[\ell']} O_{[\ell]} |\psi\rangle}{\underbrace{\langle\psi|\psi\rangle}_{= t_1^N}} \xrightarrow[\ell-\ell'\to\infty]{N\to\infty} |(T_O)^1{}_1|^2 + \mathcal{O}\left(\left(\frac{t_2}{t_1}\right)^{\ell-\ell'-1}\right)$$

If $(T_O)^1{}_1 \neq 0$: 'long-range order'

If $(T_O)^1{}_1 = 0$: 'exponential decay', $\sim e^{-|\ell-\ell'|/\xi}$

with correlation length $\xi = [\ln(t_1/t_2)]^{-1}$

2. AKLT Model

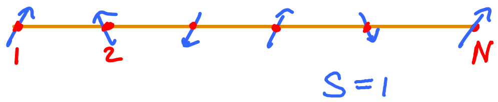
[Affleck1988], [Schollwöck2011, Sec. 4.1.5], [Tu2008]

(thanks to Hong-Hao Tu for notes!)

General remarks

- AKLT model was proposed by Affleck, Kennedy, Lieb, Tasaki in 1988.
- Previously, Haldane had predicted that S=1 Heisenberg spin chain has finite excitation gap above a unique ground state, i.e. only 'massive' excitations [Haldane1983a], [Haldane1983b].
- AKLT then constructed the first solvable, isotropic, S=1 spin chain model that exhibits a 'Haldane gap'.
- Ground state of AKLT model is an MPS of lowest non-trivial bond dimension, D=2.
- Correlation functions decay exponentially - the correlation length can be computed analytically.

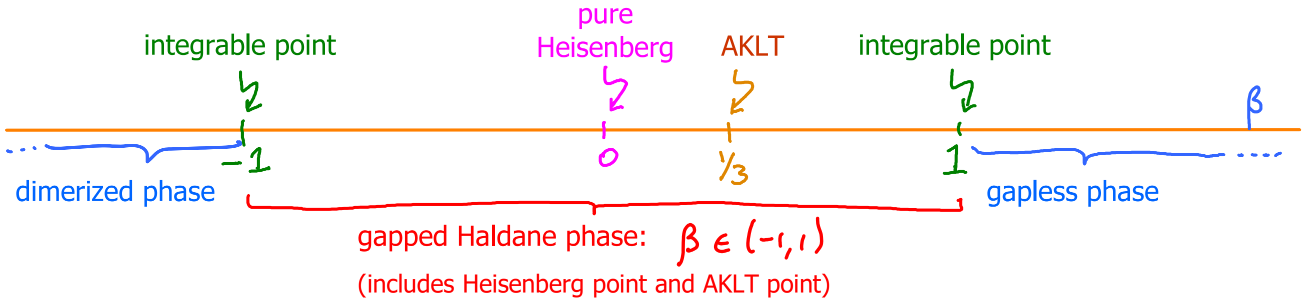
Haldane phase for S=1 spin chains



Consider bilinear-biquadratic (BB) Heisenberg model for 1D chain of spin S=1:

$$H_{BB} = \sum_{\ell=1}^{N-1} \vec{S}_\ell \cdot \vec{S}_{\ell+1} + \beta \left(\vec{S}_\ell \cdot \vec{S}_{\ell+1} \right)^2 \qquad (1)$$

Phase diagram:



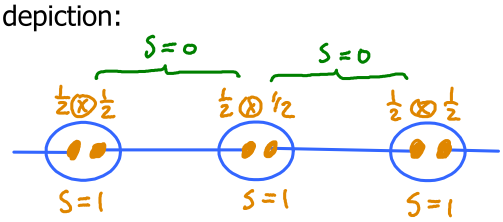
Main idea of AKLT model: $H_{AKLT} = H_{BB}(\beta = 1/3)$ (2)

is built from projectors mapping spins on neighboring sites to total spin $S^{tot}_{\ell\,\ell+1} = 2$.

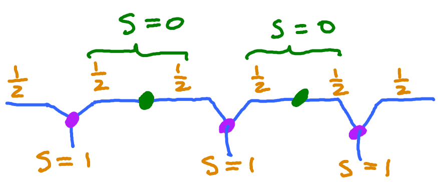
Ground state satsifies $H_{AKLT}|g\rangle = 0$. To achieve this, ground state is constructed in such a manner that spins on neighboring sites can only be coupled to $S^{tot}_{\ell,\ell+1} = 0$ or 1.

To this end, the spin-1 on each site is constructed from two auxiliary spin-1/2 degrees of freedom; One spin-1/2 each from neighboring sites is coupled to spin 0; this projects out the S=2 sector in the direct-product space of neighboring sites, ensuring that H_{AKLT} annihilates ground state.

traditional depiction:



MPS depiction: spin-1/2's live on bonds



(3)

Construction of AKLT Hamiltonian

Direct product space of spin 1 with spin 1 contains direct sum of spin 0, 1, 2:

$$\mathcal{H}_1 \otimes \mathcal{H}_1 = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad (4)$$

Projector of $\mathcal{H}_1 \otimes \mathcal{H}_1$ onto $\mathcal{H}_s$ (with $s = 0, 1, 2$)

$$P^{(s)}_{1,2} = P^{(s)}_{1,2}(\vec{S}_1, \vec{S}_2) \equiv c \prod_{s' \neq s} \left[(\vec{S}_1 + \vec{S}_2)^2 - s'(s'+1) \right] \qquad (5)$$

sites 1,2

normalization factor

yields zero when total spin = s'

Using $(\vec{S}_1 + \vec{S}_2)^2 = \underbrace{\vec{S}_1^2}_{1(1+1)} + 2\vec{S}_1 \cdot \vec{S}_2 + \underbrace{\vec{S}_2^2}_{1(1+1)} = 2\vec{S}_1 \cdot \vec{S}_2 + 4$, we find for spin-2 projector: (6)

$$P^{(2)}_{1,2} = c\left[2\vec{S}_1 \cdot \vec{S}_2 + 4 - 0(0+1)\right]\left[2\vec{S}_1 \cdot \vec{S}_2 + \underbrace{4 - 1(1+1)}_{2}\right] \qquad (7)$$

$$= c\left\{4(\vec{S}_1 \cdot \vec{S}_2)^2 + 12\,\vec{S}_1 \cdot \vec{S}_2 + 8\right\} \qquad (8)$$

Normalization is fixed by demanding that $P^{(2)}_{1,2}$ must yield 1 when acting on spin-2 subspace:

$$1 = P^{(2)}_{1,2}\Big|_{(\vec{S}_1 + \vec{S}_2)^2 = 2(2+1)} \overset{(3)}{=} c\left[2(2+1) - 0\right]\left[2(2+1) - 1(1+1)\right] \qquad (9)$$

$$\Rightarrow \quad = c\; 6 \cdot 4 \quad \Rightarrow \quad c = \frac{1}{24} \qquad (10)$$

$$\boxed{P^{(2)}_{1,2} = \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{3} \equiv P^{(2)}_{1,2}(\vec{S}_1, \vec{S}_2) = \text{projector on spin-2 subspace}} \qquad (11)$$

AKLT Hamiltonian is sum over spin-2 projectors for all neighboring pairs of spins.

$$H_{AKLT} = \sum_{\ell} P^{(2)}_{\ell,\ell+1}(\vec{S}_\ell, \vec{S}_{\ell+1}) \qquad (12)$$

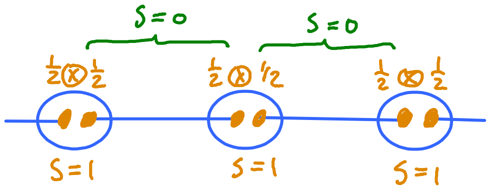
For a finite chain of N sites, use periodic boundary conditions, i.e. identify $\vec{S}_{\ell+N} = \vec{S}_\ell$.

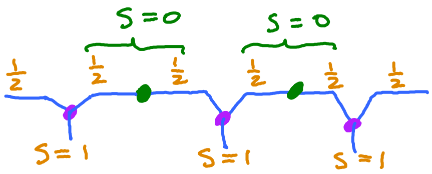
Each term is a projector, hence has only non-negative eigenvalues. Hence same is true for H_{AKLT}.

$\Rightarrow$ A state satisfying $H_{AKLT}|\psi\rangle = 0|\psi\rangle = 0$ must be a ground state!

3. AKLT ground state

MPS-III.3





On every site, represent spin 1 as symmetric combination of two auxiliary spin-1/2 degrees of freedom:

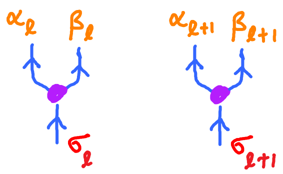
$$|S=1,\sigma\rangle \equiv |\sigma\rangle = \begin{cases} |+1\rangle = |\uparrow\rangle|\uparrow\rangle \\ |0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right) \\ |-1\rangle = |\downarrow\rangle|\downarrow\rangle \end{cases}$$

On-site projector that maps $\mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2}$ to $\mathcal{H}_1$:

$$\hat{C} = |+1\rangle\langle\uparrow|\langle\uparrow| + |0\rangle\frac{1}{\sqrt{2}}\left(\langle\uparrow|\langle\downarrow| + \langle\downarrow|\langle\uparrow|\right) + |-1\rangle\langle\downarrow|\langle\downarrow|$$

Use such a projector on every site ℓ :

$$\hat{C}_{[\ell]} = |\sigma_\ell\rangle_\ell\, C^{\sigma_\ell}_{\alpha_\ell\beta_\ell}\; {}_\ell\langle\alpha_\ell|\,{}_\ell\langle\beta_\ell|$$



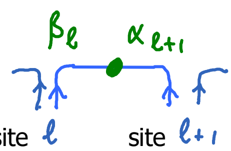
with $C^{+1} = \underset{\alpha=\beta=\uparrow}{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}$, $C^{0} = \frac{1}{\sqrt{2}}\underset{\alpha\neq\beta}{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$, $C^{-1} = \underset{\alpha=\beta=-1}{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}$ $\leftarrow$ Clebsch-Gordan Coefficients for coupling $\frac{1}{2}\otimes\frac{1}{2}\rightarrow 1$

Haldane: 'neighbors shake hands'

Now construct nearest-neighbor 'valence bonds' built from auxiliary spin-1/2 states:

$$|V\rangle_\ell = |\beta_\ell\rangle_\ell\,|\alpha_{\ell+1}\rangle_{\ell+1}\, V^{\beta_\ell\alpha_{\ell+1}} \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_\ell|\downarrow\rangle_{\ell+1} - |\downarrow\rangle_\ell|\uparrow\rangle_{\ell+1}\right)$$

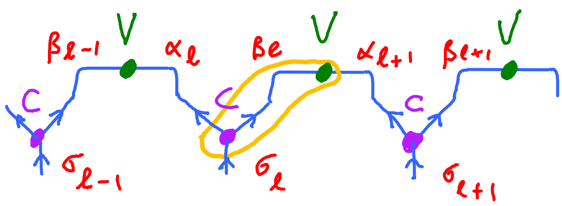
$$V = \frac{1}{\sqrt{2}}\begin{array}{c|cc} & \uparrow & \downarrow \\ \hline \uparrow & 0 & 1 \\ \downarrow & -1 & 0 \end{array}$$



Haldane: 'each site hand-shakes with its neighbors'

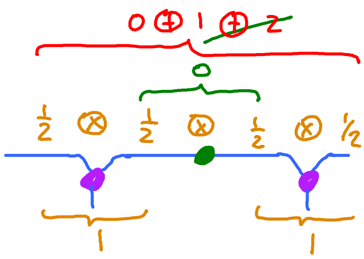
AKLT ground state = (direct product of spin-1 projectors) acting on (direct product of valence bonds):

$$|g\rangle \equiv \prod_{\otimes\ell}\hat{C}_{[\ell]}\;\prod_{\otimes\ell}|V\rangle_\ell = \cdots$$



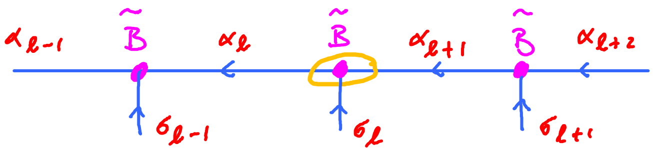
Why is this a ground state?

Coupling two auxiliary spin-1/2 to total spin 0 (valence bond)
eliminates the spin-2 sector in direct product space of two spin-1,
hence spin-2 projector in H_{AKLT} yields zero when acting on this.
(Will be checked explicitly below.)



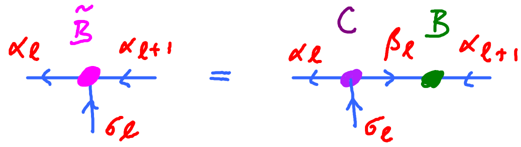
AKLT ground state is an MPS!

$$\boxed{|g\rangle = \prod_{\otimes \ell} |\sigma_\ell\rangle \, \tilde{B}_{\alpha_\ell}{}^{\sigma_\ell \alpha_{\ell+1}}}$$



with

$$\tilde{B}_{\alpha_\ell}{}^{\sigma_\ell \alpha_{\ell+1}} = C^{\sigma_\ell}{}_{\alpha_\ell \beta_\ell} \, V^{\beta_\ell \alpha_{\ell+1}}$$



Explicitly: $\sigma_\ell = +1$: $\tilde{B}^{+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\sigma_\ell = 0$: $\tilde{B}^{0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_\ell = +1$: $\tilde{B}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

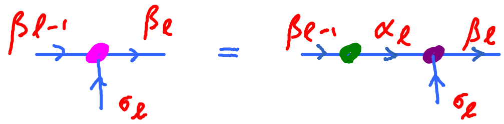
Not normalized: $\tilde{B}_\sigma \tilde{B}^{\dagger\sigma} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \frac{3}{4} \mathbb{1}$

Define right-normalized tensors, satisfying $B_\sigma B^{\dagger\sigma} = 1$: $B^\sigma \equiv \sqrt{\frac{4}{3}} \, \tilde{B}^\sigma$

$$\boxed{B^{+1} = \sqrt{\tfrac{2}{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^{0} = \tfrac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \sqrt{\tfrac{2}{3}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}}$$

[Remark: we could also have grouped B and C in opposite order, defining

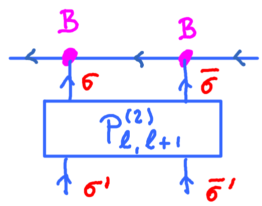
$$\tilde{A}^{\beta_{\ell-1}\sigma_\ell}{}_{\beta_\ell} = B^{\beta_{\ell-1}\alpha_\ell} \, C^{\sigma_\ell}{}_{\alpha_\ell \beta_\ell}$$



This leads to left-normalized tensors, with $A^{\pm 1} = B^{\mp 1}$, $A^{0} = B^{0}$]

<u>Exercise:</u> verify that the projector $P^{(2)}_{\ell,\ell+1}(\vec{S}_\ell, \vec{S}_{\ell+1})$

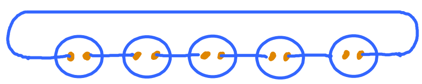
from (MPS-IV.4) yields zero when acting on sites $\ell, \ell+1$ of $|g\rangle$



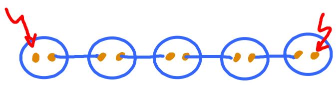
Hint: use spin-1 representation for $(\vec{S}_\ell \cdot \vec{S}_{\ell+1})^{\sigma\bar{\sigma}}{}_{\sigma'\bar{\sigma}'} = \vec{S}^{\sigma}{}_{\sigma'} \cdot \vec{S}^{\bar{\sigma}}{}_{\bar{\sigma}'}$

<u>Boundary conditions</u>

For periodic boundary conditions, Hamiltonian includes projector connecting sites 1 and N. Then ground state is unique.



For <u>open</u> boundary conditions, there are 'left-over spin-1/2' degrees of freedom at both ends of chain. Ground state is four-fold degenerate.

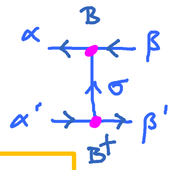


4. Transfer operator and string order parameter

MPS-III.4

(arrow directions are opposite to those of section MPS-V.1)

$$T^{a}{}_{b} = T^{\alpha'}{}_{\alpha\beta'}{}^{\beta} = B^{\dagger}{}_{\beta'\sigma}{}^{\alpha'} B_{\alpha}{}^{\sigma\beta} = \overline{B_{\alpha'}{}^{\sigma\beta'}}\, B_{\alpha}{}^{\sigma\beta}$$



$$\boxed{B^{+1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^{0} = \tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}}$$

$$T = \overline{B^{\sigma}} \otimes B^{\sigma}$$

$$= \sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hline 0 & 0 \end{array}\right)_{\sigma=1} + \tfrac{1}{\sqrt{3}}\left(\begin{array}{c|c} -1\,\tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \hline 0 & 1\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right)_{\sigma=0} + \sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 0 \\ \hline -1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \end{array}\right)_{\sigma=-1}$$

$$= \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right)$$

To compute spin-spin correlator, $C^{zz}_{\ell\ell'} \equiv \dfrac{\langle g|S^z_{[\ell]}\, S^z_{[\ell']}|g\rangle}{\langle g|g\rangle}$, we need

$$T_{S^z} = B^{\dagger}_{\sigma'}\,(S^z)^{\sigma'}{}_{\sigma}\, B^{\sigma}, \quad \text{with} \quad S^z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$= 1\cdot\sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hline 0 & 0 \end{array}\right)_{\sigma=\sigma'=1} + 0\cdot\tfrac{1}{\sqrt{3}}\left(\begin{array}{c|c} -1\,\tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ \hline 0 & 1\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right)_{\sigma=\sigma'=0} + (-1)\cdot\sqrt{\tfrac{2}{3}}\left(\begin{array}{c|c} 0 & 0 \\ \hline -1\cdot\sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \end{array}\right)_{\sigma=\sigma'=-1}$$

$$= \frac{2}{3}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)$$

Exercise

(a) Compute the eigenvalues and eigenvectors of T

(b) Show that $C^{zz}_{\ell,\ell'} \sim e^{-|\ell-\ell'|/\xi}$, with $\xi = \dfrac{1}{\ln 3}$

Remark: since the correlation length is finite, the model is gapped!

String order parameter

AKLT ground state: $|g\rangle = |\vec{\sigma}_N\rangle \, \mathrm{Tr}\left[B^{\sigma_1} B^{\sigma_2} \ldots B^{\sigma_N}\right]$ with $\sigma_\ell \in \{+1, 0, -1\}$

$$B^{+1} = \frac{2}{\sqrt{3}} \tau^+, \quad B^0 = -\frac{2}{\sqrt{3}} \tau^z, \quad B^{-1} = -\frac{2}{\sqrt{3}} \tau^-$$

with Pauli matrices $\tau^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Now, note that $B^{\pm 1} \underbrace{B^0 \ldots B^0}_{\text{string of } B^0} B^{\pm 1} = 0$

for the Pauli matrices, the operation 'raise, do nothing, raise', yields zero

Thus, all 'allowed configurations' (having non-zero coefficients) in AKLT ground state have the property that every ± 1 is followed by string of 0, then ∓ 1.

Allowed: $|\vec{\sigma}_N\rangle = \ldots \; 1 \; 0 \; 0 \; 0 \; -1 \; 0 \; 1 \; 0 \; 0 \; 0 \; 0 \; -1 \; 1 \; 0 \; 0 \; -1$

Not allowed: $|\vec{\sigma}_N\rangle = \ldots \; \underline{\underline{1}} \; 0 \; 0 \; 0 \; \underline{\underline{1}} \; 0 \; 1$ or $0 \; 0 \; \underline{\underline{-1}} \; 0 \; \underline{\underline{-1}} \; 1 \; 0$

'String order parameter' detects this property:

$$\hat{O}^{\text{string}}_{\ell\ell'} \equiv S^z_{[\ell]} \prod_{\bar{\ell}=\ell+1}^{\ell'-1} e^{i\pi S^z_{[\bar{\ell}]}} S^z_{[\ell']}$$

$$= S^z_{\ell} \; e^{i\pi S_z} \; \ldots \; e^{i\pi S_z} \; S^z_{\ell'}$$

Exercise:

Show that the ground state expectation value of string order parameter is non-zero:

$$\lim_{\ell-\ell' \to \infty} \; \lim_{N \to \infty} \langle g | \hat{O}^{\text{string}}_{\ell\ell'} | g \rangle = -\frac{4}{9}$$

Hint: first compute $T_{e^{i\pi S_z}}$

Intuitive explanation why string order parameter is nonzero:

$$|g\rangle = \sum_{\vec{\sigma}_N} |\vec{\sigma}_N\rangle \psi^{\vec{\sigma}}$$

$\ell'-1$ z

$$|g\rangle = \sum_{\sigma_N} |\sigma_N\rangle \psi^{\sigma}$$

$$C^{string}_{\ell\ell'} = \sum_{\vec\sigma} |\psi^{\vec\sigma}|^2 \langle\vec\sigma| S^z_{[\ell]}\, e^{i\pi \sum_{\bar\ell=\ell+1}^{\ell'-1} S^z_{[\bar\ell]}}\, S^z_{[\ell']} |\vec\sigma\rangle$$

For the AKLT ground state, there are six types of configurations; four of them give -1, the other two give 0:

Example configuration	$\langle\vec\sigma\| S^z_{[\ell]} \|\vec\sigma\rangle$	$\langle\vec\sigma\| S^z_{[\ell']} \|\vec\sigma\rangle$	$\langle\vec\sigma\| \sum_{\bar\ell=\ell+1}^{\ell'-1} S^z_{[\bar\ell]} \|\vec\sigma\rangle$	$\langle\vec\sigma\| S^z_{[\ell]}\, e^{i\pi\sum_{\bar\ell} S[\bar\ell]}\, S^z_{[\ell']} \|\vec\sigma\rangle$
+1 0 0 -1 0 1 0 -1 0 1	+1	+1	-1	$(+1)(+1)\cdot(-1) = -1$
-1 0 0 1 0 -1 0 1 0 -1	-1	-1	+1	$(-1)(-1)\cdot(-1) = -1$
1 0 0 0 -1 0 1 0 1 0 0 -1	+1	-1	0	$(+1)(-1)\cdot 1 = -1$
-1 0 0 1 0 -1 0 1 0 -1 1	-1	+1	0	$(-1)(+1)\cdot 1 = -1$
0 1 0 -1 1 0 -1 0 1	0			0
1 0 -1 0 1 -1 0 0 0		0		0

$$C^{string}_{\ell\ell'} = (-1)\cdot\left(\tfrac{2}{3}\right)\cdot\left(\tfrac{2}{3}\right) = -\frac{4}{9}.$$

probability to get 1 or -1 but not 0 at site ℓ

probability to get 1 or -1 but not 0 at site ℓ'