1. An example of UV completion of gauge theories: The Abelian Higgs mechanism for Proca fields.

$$\boxed{\mathcal{L}[\tilde{A}_\mu] = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{m^2}{2}\tilde{A}_\mu\tilde{A}^\mu + \frac{\xi^4}{4}\left(\tilde{A}_\mu\tilde{A}^\mu\right)^2}$$

w/ $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$

a) Stückelberg decomposition: $\boxed{\tilde{A}_\mu = A_\mu + \frac{1}{m}\partial_\mu\theta}$

gauge redundancy: $\boxed{\begin{aligned} &A_\mu \to A_\mu' = A_\mu + \frac{1}{m}\partial_\mu x \\ &\theta \to \theta' = \theta - x\end{aligned}}$

$$\hookrightarrow \tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu = \partial_\mu\left(A_\nu + \frac{1}{m}\partial_\nu\theta\right) - \partial_\nu\left(A_\mu + \frac{1}{m}\partial_\mu\theta\right)$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{m}\underbrace{\left(\partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta\right)}_{=0}$$
$$= F_{\mu\nu}$$

=0 Schwarz's theorem, partial derivatives commute

$F_{\mu\nu} \to F_{\mu\nu}' = F_{\mu\nu}$

as in standart Maxwell theory, follows again from Schwarz's thm.

$$\hookrightarrow \tilde{A}_\mu = A_\mu + \frac{1}{m}\partial_\mu\theta \to \tilde{A}_\mu' = A_\mu + \frac{1}{m}\partial_\mu x + \frac{1}{m}\partial_\mu(\theta - x)$$
$$= A_\mu + \frac{1}{m}\partial_\mu\theta + \underbrace{\frac{1}{m}\partial_\mu x - \frac{1}{m}\partial_\mu x}_{=0}$$
$$= \tilde{A}_\mu$$

So $\tilde{A}_\mu$ is gauge invariant

$$\Longrightarrow \boxed{\mathcal{L}[\tilde{A}_\mu] \to \mathcal{L}[\tilde{A}_\mu'] = \mathcal{L}[\tilde{A}_\mu]}$$

This makes sense! Gauge "symmetry" is just a redundancy and thus cannot be broken. This is beautifully seen in the Stückelberg language.

* propagating dof: set $\xi=0$ for simplicity

• eom: $\partial_\mu F^{\mu\nu} + m^2 \tilde{A}^\nu = 0 \quad | \partial_\nu$

$$\longrightarrow \underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{=0} + m^2 \partial_\nu \tilde{A}^\nu = 0$$

$$\Longrightarrow \quad \partial_\nu \tilde{A}^\nu = 0 \qquad \underline{\text{constraint}}$$

So all in al we have $\underset{\tilde{A}_\mu}{4} - \underset{\text{constraint}}{1} = 3$ propagating dof.

• This is obvious in the Stückelberg language:

A_μ: 4 dof $\xrightarrow[\text{redundancy}]{\text{gauge}}$ 2 dof

θ: 1 dof

$\Big\}$ 3 dof

In fact, the constraint is just the eom for θ.

b) <u>Rewrite Lagrangian:</u>

$$\mathcal{L}[\tilde{A}_\mu] = \frac{1}{2} A^\mu \left[(\Box + m^2)\eta_{\mu\nu} - \partial_\mu \partial_\nu \right] A^\nu$$

Propagator $\langle A^\mu(x) A^\nu(y) \rangle \equiv \Delta^{\mu\nu}(x-y)$ is per definition the Green's fct of $(\Box+m^2)\eta_{\mu\nu} - \partial_\mu\partial_\nu$.

$$\Longrightarrow \left((\Box + m^2)\eta_\mu{}^\alpha - \partial_\mu \partial^\alpha \right) \Delta_{\alpha\nu}(x) = \eta_{\mu\nu} \delta^4(x)$$

$$\overset{\mathcal{F}}{\Longrightarrow} \left((-k^2 + m^2)\eta_\mu{}^\alpha + k_\mu k^\alpha \right) \tilde{\Delta}_{\alpha\nu}(k) = \eta_{\mu\nu}$$

<u>Ansatz (Lorentz-covariance)</u>: $\tilde{\Delta}_{\alpha\nu}(k) = \eta_{\alpha\nu}\, a + k_\alpha k_\nu\, b$

w/ $a, b = ?$

<u>Plug in and compare both sides of eq.</u>

$$\left((-k^2+m^2)\eta_\mu{}^\alpha + k_\mu k^\alpha \right) \left(\eta_{\alpha\nu}\, a + k_\alpha k_\nu\, b \right)$$

$$= (-k^2+m^2)\eta_{\mu\nu}\, a + k_\mu k_\nu\, a + (-k^2+m^2) k_\mu k_\nu\, b + k^2 k_\mu k_\nu\, b$$

$$\overset{!}{=} \eta_{\mu\nu} \Longrightarrow (-k^2+m^2)\eta_{\mu\nu}\, a = \eta_{\mu\nu} \quad \leadsto \boxed{a = \frac{-1}{k^2 - m^2}}$$

$$\Longrightarrow + k_\mu k_\nu \frac{1}{k^2 - m^2} + \left((-k^2+m^2) k_{\mu\nu} + k^2 k_{\mu\nu} \right) b = 0$$

$$\leadsto \boxed{b = \frac{-a}{m^2} = \frac{1}{m^2(k^2 - m^2)}}$$

$$\Longrightarrow \boxed{\tilde{\Delta}_{\alpha\nu}(k) = \frac{1}{k^2 - m^2} \left(-\eta_{\alpha\nu} + \frac{k_\alpha k_\nu}{m^2} \right)}$$

We see that propagator is singular for $m \to 0$!!

<u>BUT</u>: Consider generating functional or the term when two sources exchange a quantum

$$\sim \int d^4x \int d^4x' \, J^\mu(x) \, \Delta_{\mu\nu}(x-x') \, J^\nu(x')$$

$$= \int \frac{d^4k}{(2\pi)^4} \, \tilde{J}^\mu(-k) \underbrace{\tilde{\Delta}_{\mu\nu}(k) \, \tilde{J}^\nu(k)}_{}$$

$$= \frac{1}{k^2 - m^2}\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right) \tilde{J}^\nu(k)$$

Look at second term. We have $k_\nu \tilde{J}^\nu(k) = 0$, due to current conservation.

$\Longrightarrow$ the singular part vanishes and we can take $m \to 0$ without a problem. (We say that the limit is smooth).

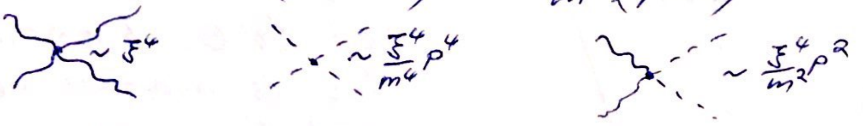
$\Longrightarrow$ Physically speaking, the longitudinal component decouples in this limit!

[As a consequence the well known photon could have a small mass, see Dvali, Adelberger, Gruzinov '03]

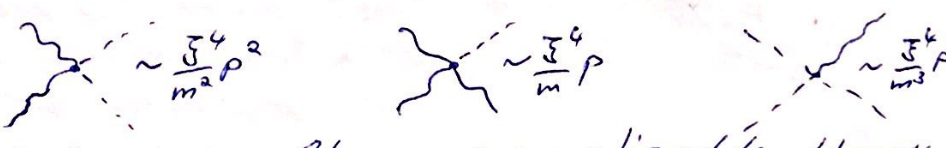
c) $$\frac{\xi^4}{4}(\tilde{A}_\mu \tilde{A}^\mu)^2 = \frac{\xi^4}{4}\left[\left(A_\mu + \frac{1}{m}\partial_\mu \chi\right)\left(A^\mu + \frac{1}{m}\partial^\mu \chi\right)\right]^2$$

$$= \frac{\xi^4}{4}\left[A_\mu A^\mu + \frac{1}{m^2}\partial_\mu \chi \partial^\mu \chi + \frac{2}{m} A_\mu \partial^\mu \chi\right]^2$$

$$= \frac{\xi^4}{4}\Big[(A_\mu A^\mu)^2 + \frac{1}{m^4}(\partial_\mu \chi \partial^\mu \chi)^2 + \frac{4}{m^2}(A_\mu \partial^\mu \chi)^2$$



$$+ \frac{2}{m^2} A_\mu A^\mu \partial_\nu \chi \partial^\nu \chi + \frac{4}{m} A_\mu A^\mu A_\nu \partial^\nu \chi + \frac{4}{m^3} \partial_\mu \chi \partial^\mu \chi A_\nu \partial^\nu \chi$$



Proca theory naively looks like a renormalizeable theory but the Stückelberg language reveals that it contains propagating dof with negative mass dimension couplings!

$\Longrightarrow$ Only an EFT, needs UV completion!

Look at "worst" coupling constant, namely the one with the highest negative mass dimension

 $\sim \frac{\xi^4}{m^4} p^4 \stackrel{!}{=} 1 \implies \boxed{p_{strong} = \frac{m^4}{\xi^4} \equiv \Lambda}$

condition for strong coupling

d) <u>Abelian Higgs model:</u>

$$\mathcal{L}[A_\mu, H] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu H)^\dagger D^\mu H - \frac{\lambda^2}{2} (H^\dagger H - v^2)^2$$

<u>gauge redundancy</u>: $\boxed{\begin{aligned} A_\mu &\to A_\mu + \tfrac{1}{g}\partial_\mu \alpha \\ H &\to e^{-i\alpha} H \end{aligned}}$

↳ $F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$ ✓

↳ $D_\mu H \to D'_\mu H' = (\partial_\mu + igA_\mu + i\partial_\mu\alpha) e^{-i\alpha} H$

$$= \partial_\mu (e^{-i\alpha} H) + (igA_\mu + i\partial_\mu \alpha) e^{-i\alpha} H$$

$$= -\cancel{i\partial_\mu \alpha}\, H + e^{-i\alpha}\partial_\mu H + (igA_\mu + \cancel{i\partial_\mu\alpha}) e^{-i\alpha} H$$

$$= e^{-i\alpha}(\partial_\mu + igA_\mu) H$$

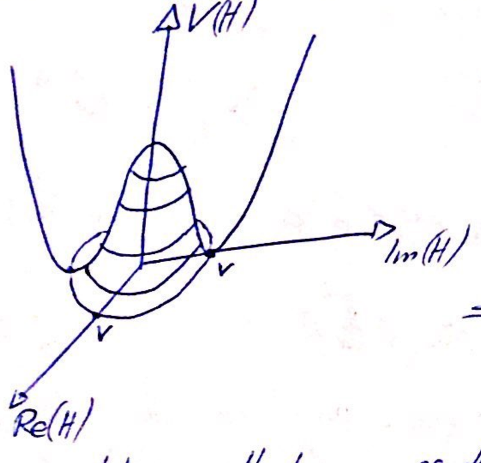
$$= e^{-i\alpha} D_\mu H$$

$\implies (D_\mu H)^\dagger D^\mu H \to (D_\mu H)^\dagger D^\mu H$ ✓

↳ $H^\dagger H \to H'^\dagger H' = (e^{-i\alpha} H)^\dagger e^{-i\alpha} H = H^\dagger e^{i\alpha} e^{-i\alpha} H = H^\dagger H$ ✓

$\implies$ <u>gauge invariant!</u> $\boxed{\mathcal{L}[A'_\mu, H'] = \mathcal{L}[A_\mu, H]}$

<u>Spontaneous Symmetry breaking:</u>



Usually, vacuum (= lowest energy state) at $\langle H \rangle = 0$. The given potential, however, has its minimum at $\langle H \rangle = v$. We say that H acquires a "vacuum expectation value" (VEV) or that the vacuum breaks the symmetry of the theory.

$\implies$ This phenomenon is known as <u>spontaneous symmetry breaking</u>.

problem: H has mass term with wrong sign ⇝ tachyonic, i.e. non physical

solution: Expand theory around the correct vacuum.

$\triangle!$ $H = (v + \frac{h(x)}{\sqrt{2}})e^{i\frac{\theta}{v}}$ need to expand around vac, which is at $\langle H\rangle = v$.

$\hookrightarrow D_\mu H = (\partial_\mu + igA_\mu)(v + \frac{h}{\sqrt{2}})e^{i\frac{\theta}{v}}$

$= \partial_\mu[(v+\frac{h}{\sqrt{2}})e^{i\frac{\theta}{v}}] + igA_\mu(v+\frac{h}{\sqrt{2}})e^{i\frac{\theta}{v}}$

$= \frac{1}{\sqrt{2}}\partial_\mu h\, e^{i\frac{\theta}{v}} + (v+\frac{h}{\sqrt{2}})i\partial_\mu\frac{\theta}{v}e^{i\frac{\theta}{v}} + igA_\mu(v+\frac{h}{\sqrt{2}})e^{i\frac{\theta}{v}}$

$= e^{i\frac{\theta}{v}}\left(\frac{1}{\sqrt{2}}\partial_\mu h + (v+\frac{h}{\sqrt{2}})i\partial_\mu\frac{\theta}{v} + igA_\mu(v+\frac{h}{\sqrt{2}})\right)$

$(D_\mu H)^\dagger D^\mu H = \left(\frac{1}{\sqrt{2}}\partial_\mu h - i(v+\frac{h}{\sqrt{2}})\partial_\mu\frac{\theta}{v} - igA_\mu(v+\frac{h}{\sqrt{2}})\right)$

$\left(\frac{1}{\sqrt{2}}\partial^\mu h + i(v+\frac{h}{\sqrt{2}})\partial^\mu\frac{\theta}{v} + igA^\mu(v+\frac{h}{\sqrt{2}})\right)$

$= \frac{1}{2}\partial_\mu h\partial^\mu h + (v+\frac{h}{\sqrt{2}})^2\partial_\mu\frac{\theta}{v}\partial^\mu\frac{\theta}{v} + g^2A_\mu A^\mu(v+\frac{h}{\sqrt{2}})^2$

$+ 2g(v+\frac{h}{\sqrt{2}})^2\partial_\mu\frac{\theta}{v}A^\mu$

$= \frac{1}{2}\partial_\mu h\partial^\mu h + g^2(v+\frac{h}{\sqrt{2}})^2(A_\mu + \frac{1}{gv}\partial_\mu\theta)^2$

$\hookrightarrow -\frac{\lambda^2}{2}(H^\dagger H - v^2)^2 = -\frac{\lambda^2}{2}(v^2 + \frac{h^2}{2} + \sqrt{2}vh - \cancel{v^2})^2$

$= -\frac{\lambda^2}{2}(\frac{h^4}{4} + 2v^2h^2 + \sqrt{2}vh^3)$

$= -\frac{\lambda^2}{8}h^4 - \lambda^2v^2h^2 - \frac{\lambda^2v}{\sqrt{2}}h^3$

$$\Longrightarrow \boxed{\begin{aligned}\mathcal{L}[h,\tilde{A}_\mu] &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2v^2\overbrace{(A_\mu + \frac{1}{gv}\partial_\mu\theta)}^{\equiv\tilde{A}_\mu}{}^2 + \frac{1}{2}\partial_\mu h\partial^\mu h \\ &\quad + g^2(\sqrt{2}vh + \frac{h^2}{2})\underbrace{(A_\mu + \frac{1}{gv}\partial_\mu\theta)}_{\equiv\tilde{A}_\mu}{}^2 - \lambda^2v^2h^2 - \frac{\lambda^2v}{\sqrt{2}}h^3 - \frac{\lambda^2}{8}h^4 \\ &= \mathcal{L}_{Proca} + \mathcal{L}_h + \mathcal{L}_{int}\end{aligned}}$$

<u>massive modes:</u> $m_A = \sqrt{2}gv$

$m_h = \sqrt{2}\lambda v$

Notes:
- ▷ The "phase field" $\theta(x)$ is the so called Goldstone boson. After spontaneous symmetry breaking of a gauge symmetry it becomes the longitudinal component of the gauge field. We say the gauge field "eats" the Goldstone.
- ▷ The expression "symmetry breaking" is quite misleading in the case of a gauge symmetry. The gauge redundancy is still there but hidden in the longitudinal dof of the gauge boson (the Stückelberg field): $h \to h$, $A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\alpha$, $\theta \to \theta - \alpha$

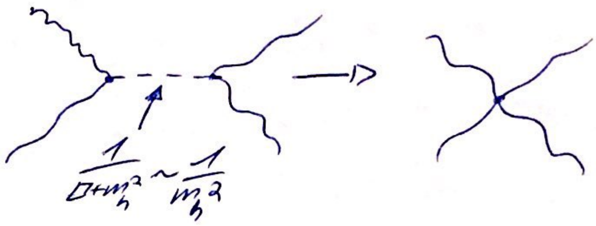
e) Calculate eom of h with lowest order interaction:

$$(\Box + m_h^2)\, h = \sqrt{2}\, g^2 v\, \tilde{A}_\mu \tilde{A}^\mu$$

$$\Longrightarrow h = \frac{\sqrt{2}\, g^2 v\, \tilde{A}_\mu \tilde{A}^\mu}{\Box + m_h^2} \overset{E \ll m_h}{=} \sqrt{2}\, g \frac{m_A}{m_h^2} \tilde{A}_\mu \tilde{A}^\mu$$

Insert in Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m_A^2}{2} \tilde{A}_\mu \tilde{A}^\mu + \underbrace{g^2 \frac{m_A^2}{m_h^2} \left(\tilde{A}_\mu \tilde{A}^\mu\right)^2} + \mathcal{O}\left(\frac{1}{m_h^2}\right)$$



Compare with Proca:

$$\frac{\xi^4}{4} \overset{!}{=} g^2 \left(\frac{m_A}{m_h}\right)^2 \Longrightarrow \boxed{\xi^2 = 2g \frac{m_A}{m_h}}$$