Chiral anomaly

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu - iQA_\mu)\psi$$

(a) • $U(1)_V$: $\psi \to e^{i\alpha}\psi$, $A_\mu \to A_\mu$

$\Rightarrow \bar{\psi} \to e^{-i\alpha}\bar{\psi} \Rightarrow \mathcal{L}_{QED}$ invariant.

Noether current: $J^\mu = \bar{\psi}\gamma^\mu\psi$ (see PS 1)

• $U(1)_A$: $\psi \to e^{i\beta\gamma^5}\psi$, $A_\mu \to A_\mu$

$$\Rightarrow \bar{\psi} \to \bar{\psi}e^{i\beta\gamma^5} \Rightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi \to \bar{\psi}\underbrace{e^{i\beta\gamma^5}\gamma^\mu}_{=\gamma^\mu e^{-i\beta\gamma^5} \text{ as } \{\gamma^5,\gamma^\mu\}=0}\partial_\mu e^{i\beta\gamma^5}\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi$$

$\Rightarrow \mathcal{L}_{QED}$ invariant.

Noether current: $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ (see PS 7)

• Now let's introduce a mass term $m\bar{\psi}\psi$, i.e. $\mathcal{L}_{QED} - m\bar{\psi}\psi$.

EOM: for ψ: $i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} - \bar{\psi}\gamma^\mu QA_\mu = 0$

for $\bar{\psi}$: $i\gamma^\mu\partial_\mu\psi - m\psi + \gamma^\mu QA_\mu\psi = 0$

Let's check the current conservation (on-shell):

$$\bullet\ \partial_\mu J^\mu = \underbrace{(\partial_\mu\bar{\psi})\gamma^\mu}_{=i(m\bar{\psi}-\bar{\psi}\gamma^\mu QA_\mu)}\psi + \bar{\psi}\underbrace{\gamma^\mu(\partial_\mu\psi)}_{=i(-m\psi+\gamma^\mu QA_\mu\psi)} = 0$$

$$\bullet\ \partial_\mu J_5^\mu = \underbrace{(\partial_\mu\bar{\psi})\gamma^\mu}_{=i(m\bar{\psi}-\bar{\psi}\gamma^\mu QA_\mu)}\gamma^5\psi + \bar{\psi}\underbrace{\gamma^\mu\gamma^5}_{=-\gamma^5\gamma^\mu}(\partial_\mu\psi) =$$

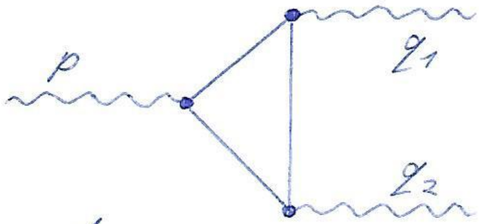
$$= im\bar{\psi}\gamma^5\psi - i\bar{\psi}\gamma^\mu QA_\mu\gamma^5\psi - \bar{\psi}\gamma^5 i(-m\psi + \gamma^\mu QA_\mu\psi) =$$

$$= 2im\bar{\psi}\gamma^5\psi \longrightarrow 0 \text{ for } m \to 0.$$

$\Rightarrow$ So classically J_5^μ is conserved, if $m=0$.
We will next show that this is not true quantum mechanically.

(b) • Gauge anomalies are associated with triangular amplitudes with massless gauge bosons in the external legs (in this case photons)

• Since external photons can be created from the vacuum by the interaction term



$H_{int} = -\int d^3x\, \bar{\psi}\gamma^\mu Q A_\mu \psi = -\int d^3x\, Q J^\mu A_\mu$, the above amplitude is given (schematically) as

$$\delta^{(4)}(p-q_1-q_2)\underbrace{i\mathcal{M}}_{\equiv \mathcal{M}^{\alpha\mu\nu}\varepsilon_\alpha \varepsilon^*_\mu \varepsilon^*_\nu} \sim \left(\int d^4x\right)^3 \underbrace{\langle q_1, q_2 |\, H^3_{int}\, | p\rangle}_{\to\, \varepsilon_\alpha e^{-ipx}\, \varepsilon^*_\mu e^{iq_1 y}\, \varepsilon^*_\nu e^{iq_2 z}}$$

⇒ Hence we can look at the correlation fct.

$$\langle \Omega | T\{J^\alpha_5(x) J^\mu(y) J^\nu(z)\} | \Omega\rangle \equiv \langle J^\alpha_5(x) J^\mu(y) J^\nu(z)\rangle$$

(assuming that the initial photon couples to the axial current J^α_5; we consider $m=0$ now)

$$\Rightarrow \langle J^\alpha_5(x) J^\mu(y) J^\nu(z)\rangle = \langle \bar{\psi}_x \gamma^\alpha\gamma^5 \psi_x \bar{\psi}_y \gamma^\mu \psi_y \bar{\psi}_z \gamma^\nu \psi_z\rangle =$$

= (keeping only terms which contribute to triangular diagram) $= \bar{\psi}_x \gamma^\alpha \gamma^5 \psi_x \bar{\psi}_y \gamma^\mu \psi_y \bar{\psi}_z \gamma^\nu \psi_z =$ two contributions

$$= -\mathrm{tr}\left[S_F(z-x)\gamma^\alpha\gamma^5 S_F(x-y)\gamma^\mu S_F(y-z)\gamma^\nu\right] - \mathrm{tr}\left[S_F(y-x)\gamma^\alpha\gamma^5 S_F(x-z)\gamma^\nu S_F(z-y)\gamma^\mu\right]$$

with the Dirac propagator $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \frac{i\cdot e^{-ik(x-y)}}{\not{k}-m}$

$$\Rightarrow \int d^4x\, d^4y\, d^4z\, e^{-ipx} e^{iq_1 y} e^{iq_2 z} \langle J^\alpha_5(x) J^\mu(y) J^\nu(z)\rangle =$$

$$= -\int d^4k_1\, d^4k_2\, d^4k_3\, \delta^{(4)}(p-k_1+k_2)\,\delta^{(4)}(q_1+k_2-k_3)\,\delta^{(4)}(q_2-k_1+k_3)\,\cdot$$

$$\cdot\, \mathrm{tr}\left[\frac{i}{\not{k}_1-m}\gamma^\alpha\gamma^5 \frac{i}{\not{k}_2-m}\gamma^\mu \frac{i}{\not{k}_3-m}\gamma^\nu\right] -$$

$$-\int d^4k_1\, d^4k_2\, d^4k_3\, \delta^{(4)}(p-k_1+k_2)\,\delta^{(4)}(q_1-k_1+k_3)\,\delta^{(4)}(q_2+k_2-k_3)\,\cdot$$

$$\cdot\, \mathrm{tr}\left[\frac{i}{\not{k}_1-m}\gamma^\alpha\gamma^5 \frac{i}{\not{k}_2-m}\gamma^\nu \frac{i}{\not{k}_3-m}\gamma^\mu\right] \overset{(m=0)}{=}$$

$$= -\int d^4k\, \delta^{(4)}(p-q_1-q_2)\, \mathrm{tr}\left[\frac{i}{\not k}\gamma^\alpha\gamma^5 \frac{i}{\not k-\not q_1-\not q_2}\gamma^\mu \frac{i}{\not k-\not q_2}\gamma^\nu + \right.$$

$$\left. + \frac{i}{\not k+\not q_1+\not q_2}\gamma^\alpha\gamma^5\frac{i}{\not k}\gamma^\nu\frac{i}{\not k+\not q_2}\gamma^\mu\right]$$

Let's shift the loop-momentum k such that the above expression reflects the momentum assignments of the diagrams. $k \to k+q_2$ in the first term and $k\to k-q_2$ in the second.

$$\Rightarrow \int d^4x\, d^4y\, d^4z\, e^{-ipx} e^{iq_1y} e^{iq_2z} \langle J_5^\alpha(x) J^\mu(y) J^\nu(z)\rangle =$$

$$= (2\pi)^4 \delta^{(4)}(p-q_1-q_2)\, i\mathcal{M}_5^{\alpha\mu\nu}(p, q_1, q_2)$$

with $i\mathcal{M}_5^{\alpha\mu\nu}(p,q_1,q_2) \equiv -\int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\gamma^\mu \frac{i}{\not k}\gamma^\nu \frac{i}{\not k+\not q_2}\gamma^\alpha\gamma^5\frac{i}{\not k-\not q_1} + \right.$

$$\left. + \gamma^\nu\frac{i}{\not k}\gamma^\mu\frac{i}{\not k+\not q_1}\gamma^\alpha\gamma^5\frac{i}{\not k-\not q_2}\right]$$

(c) We do not want to calculate $i\mathcal{M}_5^{\alpha\mu\nu}$ itself, but rather use it to check, if J_5^α and J^μ are conserved in the quantum theory.

$\Rightarrow$ For the axial current we have to check

$$p_\alpha \mathcal{M}_5^{\alpha\mu\nu} \sim \left\langle \left(\frac{\partial}{\partial x^\alpha} J_5^\alpha(x)\right) J^\mu(y) J^\nu(z)\right\rangle \overset{?}{=} 0$$

and for the vector current

$$q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu} \sim \left\langle J_5^\alpha(x) \left(\frac{\partial}{\partial y^\mu} J^\mu(y)\right) J^\nu(z)\right\rangle \overset{?}{=} 0$$

Axial: $p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = \int\frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left[\frac{\gamma^\mu \not k \gamma^\nu(\not k+\not q_2)}{k^2(k+q_2)^2(k-q_1)^2}\not p\gamma^5(\not k-\not q_1) + \begin{pmatrix}\mu \leftrightarrow \nu\\ 1\leftrightarrow 2\end{pmatrix}\right]$

Note that this integral goes like $\sim \int \frac{d^4k}{k^3}$ and is therefore superficially linearly divergent.

Now let's simplify it by using momentum-conservation:

$p = q_1 + q_2$

$$\Rightarrow \not p\gamma^5 = (\not q_1+\not q_2)\gamma^5 = (\not q_1 - \not k + \not k + \not q_2)\gamma^5 = \gamma^5(\not k - \not q_1) + (\not k+\not q_2)\gamma^5$$

$$\Rightarrow p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \, tr\left[\frac{\overbrace{\gamma^\mu \not{k} \gamma^\nu (\not{k}+\not{q}_2)\gamma^5}^{=(1)}}{k^2(k+q_2)^2} + \frac{\overbrace{\gamma^\mu \not{k} \gamma^\nu \gamma^5 (\not{k}-\not{q}_1)}^{=(2)}}{k^2(k-q_1)^2} + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix} \right]$$

- $tr[(1)] = -4i\varepsilon^{\mu\alpha\nu\beta} k_\alpha (k_\beta + q_{2,\beta}) = -4i\varepsilon^{\mu\alpha\nu\beta} k_\alpha q_{2,\beta}$
- $tr[(2)] = tr[(\not{k}-\not{q}_1)\gamma^\mu \not{k} \gamma^\nu \gamma^5] = -4i\varepsilon^{\alpha\mu\beta\nu}(k_\alpha - q_{1,\alpha})k_\beta = 4i\varepsilon^{\alpha\mu\beta\nu} q_{1,\alpha} k_\beta$

$$\Rightarrow p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = 4i\varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \left(\frac{k_\alpha q_{2,\beta}}{k^2(k+q_2)^2} + \frac{k_\alpha q_{1,\beta}}{k^2(k-q_1)^2} \right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix}$$

The term $\int \frac{d^4k}{(2\pi)^4} \frac{k^\alpha q_2^\beta}{k^2(k+q_2)^2}$ is a Lorentz tensor that only depends on q_2, so by Lorentz invariance it has to be proportional to $q_2^\alpha q_2^\beta$, but this gives zero when contracted with $\varepsilon^{\mu\nu\alpha\beta}$. The same goes for the 2nd term.

↳ So it looks as if $p_\alpha \mathcal{M}_5^{\alpha\mu\nu} \stackrel{?}{=} 0$

$\Rightarrow$ Axial current conserved even quantum mechanically?

Vector: $q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \, tr\left[\frac{\not{q}_1 \not{k} \gamma^\nu (\not{k}+\not{q}_2)\gamma^\alpha\gamma^5(\not{k}-\not{q}_1)}{k^2(k+q_2)^2(k-q_1)^2} + \frac{\gamma^\nu \not{k} \not{q}_1 (\not{k}+\not{q}_1)\gamma^\alpha\gamma^5(\not{k}-\not{q}_2)}{k^2(k+q_1)^2(k-q_2)^2} \right]$

(again, linearly divergent)

Let's simplify by writing $\not{q}_1 = \not{k} - (\not{k}-\not{q}_1)$ in the first term and $\not{q}_1 = (\not{k}+\not{q}_1) - \not{k}$ in the second and use the cyclic property of the trace:

$$q_{1,\mu}\mathcal{M}_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \, tr\left[\frac{\overbrace{\gamma^\nu(\not{k}+\not{q}_2)\gamma^\alpha\gamma^5(\not{k}-\not{q}_1)}^{=\boxed{1}}}{(k+q_2)^2(k-q_1)^2} - \frac{\overbrace{\not{k}\gamma^\nu(\not{k}+\not{q}_2)\gamma^\alpha\gamma^5}^{=\boxed{2}}}{k^2(k+q_2)^2} + \frac{\overbrace{\gamma^\nu \not{k} \gamma^\alpha \gamma^5 (\not{k}-\not{q}_2)}^{=\boxed{3}}}{k^2(k-q_2)^2} - \frac{\overbrace{\gamma^\nu(\not{k}+\not{q}_1)\gamma^\alpha\gamma^5(\not{k}-\not{q}_2)}^{=\boxed{4}}}{(k+q_1)^2(k-q_2)^2} \right]$$

- $tr[\boxed{1}] = -4i\varepsilon^{\mu\nu\beta\alpha}(k_\mu - q_{1,\mu})(k_\beta + q_{2,\beta})$
- $tr[\boxed{2}] = -4i\varepsilon^{\mu\nu\beta\alpha} k_\mu (k_\beta + q_{2,\beta}) = -4i\varepsilon^{\mu\nu\beta\alpha} k_\mu q_{2,\beta}$
- $tr[\boxed{3}] = -4i\varepsilon^{\mu\nu\beta\alpha}(k_\mu - q_{2,\mu})k_\beta = 4i\varepsilon^{\mu\nu\beta\alpha} q_{2,\mu} k_\beta = tr[\boxed{2}]$
- $tr[\boxed{4}] = -4i\varepsilon^{\mu\nu\beta\alpha}(k_\mu - q_{2,\mu})(k_\beta + q_{1,\beta})$

Parts 2 and 3 vanish for the same reason as before (Lorentz invariance).

$$\Rightarrow q_{3,\mu} M_5^{\alpha\mu\nu} = -4i\varepsilon^{\alpha\nu\mu\beta} \int \frac{d^4k}{(2\pi)^4} \left[\frac{(k_\mu - q_{3,\mu})(k_\beta + q_{2,\beta})}{(k+q_2)^2 (k-q_1)^2} - \frac{(k_\mu - q_{2,\mu})(k_\beta + q_{3,\beta})}{(k+q_1)^2 (k-q_2)^2} \right]$$

We would like to shift $k \to k+q_1$ in the first term and $k \to k+q_2$ in the second, so that both terms cancel each other. However, the integral is linearly divergent, so we have to be very careful with shifting the loop-momentum! (The quadratically divergent part is prop. to $k_\mu k_\beta$ and vanishes due to the Levi-Civita tensor).

Consider the integral

$$\Delta^\alpha(a^\mu) = \int \frac{d^4k}{(2\pi)^4} \left[F^\alpha(k+a) - F^\alpha(k) \right]$$

where the first integrand is shifted by a^μ w.r.t. the second.

Let's assume that we can Wick rotate: $k^0 \to i k_E^0$.

$$\Rightarrow \Delta^\alpha(a^\mu) = i \int \frac{d^4k_E}{(2\pi)^4} \left[F^\alpha(k_E+a) - F^\alpha(k_E) \right] \overset{\substack{\text{Taylor} \\ \text{expansion}}}{=}$$

$$= i \int \frac{d^4k_E}{(2\pi)^4} \left[a^\mu \frac{\partial F^\alpha(k_E)}{\partial k_E^\mu} + \frac{1}{2} a^\mu a^\nu \frac{\partial^2 F^\alpha(k_E)}{\partial k_E^\mu \partial k_E^\nu} + \dots \right]$$

Since we take the integral to be linearly divergent,

$$\lim_{k_E \to \infty} F^\alpha(k_E) = A \frac{k_E^\alpha}{k_E^4}, \quad A \text{ is const. in } k_E.$$

All terms with $O(a^2)$ vanish after using Gauss' theorem, so we can drop them.

$$\Rightarrow \Delta^\alpha(a^\mu) \overset{\substack{\text{Gauss'} \\ }}{\underset{\text{theorem}}{=}} \frac{i}{(2\pi)^4} a^\mu \oint_{k_E \to \infty} d^3S_\mu \, F^\alpha(k_E) \quad \text{with the surface}$$

element $d^3S_\mu = k_E^2 \, k_{E,\mu} \, d\Omega_4$.

$$\Rightarrow \Delta^\alpha(a^\mu) = \frac{i}{(2\pi)^4} a_\mu A \oint d\Omega_4 \frac{k_E^\mu k_E^\alpha}{k_E^2}$$

unit-vector

Recall that in 4 (Euclidean) dimensions $\oint d\Omega_4 \, n^\mu n^\alpha = \underbrace{\Omega_4}_{=2\pi^2} \frac{1}{4} \delta^{\mu\alpha}$

$$\Rightarrow \Delta^\alpha(a^\mu) = \frac{i}{32\pi^2} A a^\alpha$$

So we see that a linearly divergent integral, which would vanish for a certain shift a^μ, is finite and proportional to the necessary shift.

Let's go back to our final expression of $q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu}$: the second term has the form

$$\begin{pmatrix} \text{term prop. to } k_\mu k_\beta \text{ that} \\ \text{vanishes due to } \varepsilon^{\mu\nu\alpha\beta} \end{pmatrix} + \frac{k_\mu q_{1,\beta} - k_\beta q_{2,\mu}}{(k+q_2)^2(k-q_1)^2} + \mathcal{O}\left(\frac{1}{k^4}\right) \overset{\text{using Levi-Civita}}{=}$$

$$= (q_{1,\beta} + q_{2,\beta}) \underbrace{\frac{k_\mu}{(k+q_2)^2(k-q_1)^2}}_{= F_\mu(k)} \quad , \; A = 1.$$

The first term of $q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu}$ is then $(q_{1,\beta} + q_{2,\beta}) F_\mu(k+a)$ with $a^\mu = q_2^\mu - q_1^\mu$.

$$\Rightarrow q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu} = \frac{1}{8\pi^2} \varepsilon^{\alpha\nu\mu\beta} (q_{1,\beta} + q_{2,\beta})(q_{2,\mu} - q_{1,\mu}) = \frac{1}{4\pi^2} \varepsilon^{\alpha\nu\mu\beta} q_{1,\beta} q_{2,\mu} \neq 0$$

$\Rightarrow$ Vector current not conserved?

(d) Remember that the choice of the loop-momentum k was arbitrary from the beginning. So the fact that we just found that the various contractions of $\mathcal{M}_5^{\alpha\mu\nu}$ give a finite, but shift-dependent result, shows us that the result depends on our choice of momentum routing in the loops (as long as we stick to one choice and don't change it for different contractions of $\mathcal{M}_5^{\alpha\mu\nu}$).

Now let's use the most general choice

$k^\mu \to k^\mu + b_1 q_1^\mu + b_2 q_2^\mu$ (for the first diagram).

Since we want the 2nd diagram to be symm. under the exchange of $q_1 \leftrightarrow q_2$ (photons are bosons), we take

$k^\mu \to k^\mu + b_2 q_1^\mu + b_1 q_2^\mu$ (for the second diagram).

• Vector: recall that the term $\dfrac{(k_\mu - q_{1,\mu})(k_\beta + q_{2,\beta})}{(k-q_1)^2(k+q_2)^2}$ in $q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu}$ corresponds to the first diagram, while $\dfrac{(k_\mu - q_{2,\mu})(k_\beta + q_{1,\beta})}{(k-q_2)^2(k+q_1)^2}$ corresponds to the second, so making the above replacements for k, leads to $F^\mu(k+a) - F^\mu(k)$ with $a^\mu = (1+b_2-b_1)(q_2^\mu - q_1^\mu)$

$$\Rightarrow q_{1,\mu} \mathcal{M}_5^{\alpha\mu\nu} = \frac{1}{4\pi^2} \varepsilon^{\alpha\nu\mu\beta} q_{1,\beta} q_{2,\mu} (1+b_2-b_1)$$

• Axial: now we are forced to make the same replacement (momentum routing) in the axial calulation. Our final expression was

$$p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = 4i\varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \Bigg[\underbrace{\frac{k_\alpha q_{2,\beta}}{k^2(k+q_2)^2}}_{=\mathrm{I}} + \underbrace{\frac{k_\alpha q_{1,\beta}}{k^2(k-q_1)^2}}_{=\mathrm{II}} - \Bigg(\underbrace{\frac{k_\alpha q_{1,\beta}}{k^2(k+q_1)^2}}_{=\mathrm{III}} + \underbrace{\frac{k_\alpha q_{2,\beta}}{k^2(k-q_2)^2}}_{=\mathrm{IV}} \Bigg) \Bigg]$$

$$\mathrm{I} - \mathrm{IV} \xrightarrow{\text{shift}} \frac{(k+b_1q_1+b_2q_2)_\alpha q_{2,\beta}}{(k+b_1q_1+b_2q_2)^2(k+b_1q_1+(b_2+1)q)^2} - \frac{(k+b_2q_1+b_1q_2)_\alpha q_{2,\beta}}{(k+b_2q_1+b_1q_2)^2(k+b_2q_1+(b_1-1)q_2)^2} =$$

= (dropping terms $q_{2,\alpha} q_{2,\beta}$ in the numerator) =

$$= q_{2,\beta}\left[F_{1,\alpha}(k+a_1) - F_{1,\alpha}(k)\right]$$

with $F_{1,\alpha}(k) = \dfrac{(k+b_2q_1)_\alpha}{(k+b_2q_1+b_1q_2)^2(k+b_2q_1+(b_1-1)q_2)^2}$

and $a_1^\mu = -(b_2-b_1)q_1^\mu + (1+b_2-b_1)q_2^\mu$

$$\mathrm{II} - \mathrm{III} \xrightarrow{\text{shift}} \frac{(k+b_1q_1+b_2q_2)_\alpha q_{1,\beta}}{(k+b_1q_1+b_2q_2)^2(k+(b_1-1)q_1+b_2q_2)^2} -$$

$$- \frac{(k+b_2q_1+b_1q_2)_\alpha q_{1,\beta}}{(k+b_2q_1+b_1q_2)^2(k+(b_2+1)q_1+b_1q_2)^2} = q_{1,\beta}\left[F_{2,\alpha}(k+a_2) - F_{2,\alpha}(k)\right]$$

with $F_{2,\alpha}(k) = \dfrac{(k+b_1q_2)_\alpha}{(k+b_2q_1+b_1q_2)^2(k+(b_2+1)q_1+b_1q_2)^2}$

and $a_2^\mu = -(b_2-b_1+1)q_1^\mu + (b_2-b_1)q_2^\mu$

$$\Rightarrow p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = 4i\,\varepsilon^{\mu\nu\alpha\beta}\underbrace{\left(q_{2,\beta}\,\Delta_{1,\alpha}(a_1) + q_{1,\beta}\,\Delta_{2,\alpha}(a_2)\right)}_{=\frac{i}{32\pi^2}(-2)(b_2-b_1)\,q_{1,\alpha}\,q_{2,\beta}} =$$

(again, dropping $q_{1,\alpha} q_{1,\beta}$ etc.)

$$= \frac{1}{4\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}(b_2-b_1)\,q_{1,\alpha}\,q_{2,\beta}$$

$\Rightarrow$ We see that for $b_1 = b_2$ we recover our previous results for the vector and axial currents.

However, we should use the choice, which is compatible with a non-zero fermion mass, $b_1 - b_2 = 1$.

$\Rightarrow q_{1,\mu}\mathcal{M}_5^{\alpha\mu\nu} = 0$ (vector current conserved)

$p_\alpha \mathcal{M}_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}\,q_{1,\beta}\,q_{2,\alpha}$ (axial current not conserved)

(e) $\mathcal{L} = -\frac{1}{4}F^2 + i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_L\gamma^\mu Q_L A_\mu \psi_L$

with $\psi_L = P_L\psi = \frac{1}{2}(\mathbb{1}-\gamma^5)$

$U(1)_L:\ \psi_L \to e^{-i\beta}\psi_L$

$\Rightarrow$ current: $J_L^\mu = \bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi}\gamma^\mu P_L\psi$

$$\Rightarrow \mathcal{M}_L^{\alpha\mu\nu} = \int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{\gamma^\mu P_L \not{k}\gamma^\nu P_L(\not{k}+\not{q}_2)\gamma^\alpha P_L(\not{k}-\not{q}_1)}{k^2(k+q_2)^2(k-q_1)^2} + \begin{pmatrix}\mu\leftrightarrow\nu\\ 1\leftrightarrow 2\end{pmatrix}\right]$$

$$\Rightarrow \mathcal{M}_L^{\alpha\mu\nu} = \frac{1}{2}\left(\mathcal{M}_V^{\alpha\mu\nu} - \mathcal{M}_5^{\alpha\mu\nu}\right)$$

$\sim \langle J^\alpha J^\mu J^\nu\rangle$ $\qquad\qquad \sim \langle J_5^\alpha J^\mu J^\nu\rangle$

It is straightforward to check that $\mathcal{M}_V^{\alpha\mu\nu}$ vanishes for <u>any</u> contraction (Ward identity).

So we know from our result of $\mathcal{M}_5^{\alpha\mu\nu}$ that either

$p_\alpha \mathcal{M}_L^{\alpha\mu\nu} \neq 0$ or $q_{1,\mu}\mathcal{M}_L^{\alpha\mu\nu} \neq 0$

$\Rightarrow$ either $\partial_\alpha\langle J_L^\alpha J_L^\mu J_L^\nu\rangle \neq 0$ or $\partial_\mu\langle J_L^\alpha J_L^\mu J_L^\nu\rangle \neq 0$.

$\Rightarrow$ So the symm. $U(1)_L$ must be anomalous and we cannot couple the photon to a conserved current consistently in a theory with just a left-handed (or right-handed) Weyl fermion.

(f) $$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi} i \not{\partial} P_L \psi + \bar{\psi} i \not{\partial} P_R \psi + A_\mu \underbrace{\left(\bar{\psi} Q_L \gamma^\mu P_L \psi + \bar{\psi} Q_R \gamma^\mu P_R \psi\right)}_{\equiv J^\mu_{mix} = Q_L J^\mu_L + Q_R J^\mu_R}$$

Now the massless gauge boson A_μ couples to the current J^μ_{mix}, which is due to the symm.

$\psi_L \to e^{i\alpha Q_L} \psi_L$

$\psi_R \to e^{i\alpha Q_R} \psi_R$

Now the triangular diagram, which contributes to $\langle J^\alpha_{mix} J^\mu_{mix} J^\nu_{mix} \rangle$ has either ψ_L running in the loop or ψ_R, but they don't mix!

So there is no contribution like $\langle J^\alpha_L J^\mu_L J^\nu_R \rangle Q_L^2 Q_R$, etc.

$$\Rightarrow \mathcal{M}^{\alpha\mu\nu}_{mix} = Q_L^3 \mathcal{M}^{\alpha\mu\nu}_L + Q_R^3 \mathcal{M}^{\alpha\mu\nu}_R \underset{\text{dropping } \mathcal{M}^{\alpha\mu\nu}_V}{=} \frac{1}{2}\left(Q_R^3 - Q_L^3\right) \mathcal{M}^{\alpha\mu\nu}_5 .$$