① The (massive) neutrino

(a) There are two different ways to write down a Lorentz invariant mass term for fermions:

Dirac mass: $m_D \bar{\psi}_L \psi_R + H.c.$

Majorana mass: $m_M \bar{\psi}_L^c \psi_L + H.c.$

(or $m_M \bar{\psi}_R^c \psi_R + H.c.$)

with $\psi_L^c = C \bar{\psi}_L^T = i\gamma^2 \psi_L^* \equiv (\psi^c)_R$ $\quad \rightarrow m_M \bar{\psi}_L^c \psi_L = m_M \psi_L^T C \psi_L$

- The Majorana mass term cannot describe charged particles, since it violates the conservation of charge.

↳ So the neutrino is the only SM fermion, which could, in principle, have a Dirac mass and/or a Majorana mass. Note that in the latter case, the neutrino cannot carry a lepton number, so lepton number would be violated in the SM!

(b) For the Dirac mass we can use, analogously to the quark case, the Higgs doublet with opposite hypercharge, $\tilde{H} = i\sigma^2 H^*$:

Dirac: $-\Lambda_{ij}^{(\nu)} \bar{E}_L^i \tilde{H} \nu_R^j + H.c.$

Majorana: $-M_{ij} \bar{\nu}_R^{c\,i} \nu_R^j + H.c.$

↳ Note that we cannot write down a Majorana mass term for ν_L, because it transforms under $SU(2) \times U(1)_Y$ (it carries quantum numbers).

(c) • Let's first neglect the Majorana mass term.

- We can now again (like in ex. 9.1) rotate to the mass basis via

$e_R \rightarrow K_e e_R,\ e_L \rightarrow U_e e_L,\ \nu_R \rightarrow K_\nu \nu_R,\ \nu_L \rightarrow U_\nu \nu_L.$

↳ Now the mass terms of the leptons are diagonal, however the (charged) interaction term

$\frac{g}{\sqrt{2}}\left[\bar{\nu}_L^i \cancel{W}^+ e_L^i + \bar{e}_L^i \cancel{W}^- \nu_L^i\right]$ becomes

$$\xrightarrow[\text{basis}]{\text{mass}} \frac{g}{\sqrt{2}}\left[\underbrace{\left(U_\nu^\dagger U_e\right)}_{\equiv P}{}_{ij}\, \bar{\nu}_L^i \cancel{W}^+ e_L^j + \underbrace{\left(U_e^\dagger U_\nu\right)}_{= P^\dagger}{}_{ij}\, \bar{e}_L^i \cancel{W}^- \nu_L^j\right],$$

where $P_{ij} = \left(U_\nu^\dagger U_e\right)_{ij}$ is the PMNS-matrix.

(d) Let's denote the neutrino flavors (weak basis) by ν_α and the mass eigenstates by ν_i, where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$.

$\Rightarrow \nu_\alpha = P_{\alpha i} \nu_i$ (sum over i)

Let's consider a beam of neutrinos, which at time $t=0$ get produced as a weak eigenstate $|\nu_\alpha(0)\rangle$.

We can now expand this in mass eigenstates as

$|\nu_\alpha(0)\rangle = P_{\alpha i}|\nu_i\rangle$.

Let's assume that all neutrinos in the beam have the same momentum, then $E_i^2 = \vec{p}^{\,2} + m_i^2$.

Now the different energy eigenvalues E_i of the energy eigenstates $|\nu_i\rangle$ will cause the state $|\nu_\alpha(0)\rangle$ to evolve:

$$|\nu_\alpha(t)\rangle = e^{-iHt}|\nu_\alpha(0)\rangle = P_{\alpha i}\, e^{-iE_i t}|\nu_i\rangle.$$

↳ So the probability of finding ν_β in a beam of (initially) ν_α after time t is:

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta(0)|\nu_\alpha(t)\rangle|^2 = |\langle \nu_j | P_{\beta j}^* P_{\alpha i}\, e^{-iE_i t}|\nu_i\rangle|^2 =$$

$$= |P_{\beta i}^* P_{\alpha i}\, e^{-iE_i t}|^2 = \left(\sum_i P_{\beta i}^* P_{\alpha i}\, e^{-iE_i t}\right)\left(\sum_j P_{\beta j} P_{\alpha j}^*\, e^{iE_j t}\right) =$$

$$= \sum_i |P_{\alpha i}|^2 |P_{\beta i}|^2 + \sum_{i \neq j} P_{\alpha i} P_{\beta i}^* P_{\alpha j}^* P_{\beta j}\, e^{-i(E_i - E_j)t}$$

For ultra-relativistic neutrinos $(\overbrace{\vec{p}^{\,2}}^{\equiv p^2} \gg m_i^2)$:

$$E_i = (p^2 + m_i^2)^{1/2} = p\left(1 + \frac{m_i^2}{p^2}\right)^{1/2} \simeq p + \frac{m_i^2}{2p} \quad \Rightarrow E_i - E_j \simeq \frac{1}{2p}\left(m_i^2 - m_j^2\right)$$

Let's define the oscillation length

$$l_{ij} = \frac{2\pi}{|E_i - E_j|} \simeq \frac{4\pi p}{|m_i^2 - m_j^2|} \quad \text{and use} \quad t \simeq x.$$

To illustrate, consider just two neutrino generations with $[P_{\alpha i}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow P_{\nu_e \to \nu_\mu}(x) = 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta\left(e^{-i(E_1-E_2)x} + e^{-i(E_2-E_1)x}\right) =$$

$$= 2\sin^2\theta\cos^2\theta\left(1 - \cos\left[(E_1-E_2)x\right]\right) =$$

$$= \sin^2(2\theta)\sin^2\left(\frac{\pi x}{l_{12}}\right)$$

$\hookrightarrow$ We see that for a given mixing angle $\theta > 0$, the probability of finding a different neutrino is maximal at $x = \frac{l_{12}}{2}$. After l_{12} the neutrino will oscillate back to its original flavor. For a given momentum, the oscillation length is sensitive only to the difference of squares of neutrino masses, $l_{ij} \sim \frac{1}{|m_i^2 - m_j^2|}$. [See additional note on next page: *].

(e) Let's assume real masses for simplicity.

$$\mathcal{L}^{mass} = -m\,\bar{\nu}_L \nu_R - \frac{M}{2}\bar{\nu}_R^c \nu_R + H.c.$$

Let's define $N_L \equiv \nu_R^c = C\bar{\nu}_R^T \quad \Rightarrow \nu_R = C\bar{N}_L^T$.

Then $\bar{\nu}_R \nu_L = N_L^T C \nu_L$ and $(\bar{\nu}_R^c \nu_R)^\dagger = \bar{\nu}_R \nu_R^c = N_L^T C N_L$.

$$\Rightarrow \mathcal{L}^{mass} = -m\, N_L^T C \nu_L - \frac{M}{2} N_L^T C N_L + H.c. = -\frac{1}{2} n_L^T C \mathcal{M} n_L + H.c.$$

with $n_L = \begin{pmatrix} \nu_L \\ N_L \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$, where we used

$$\frac{1}{2} N_L^T C \nu_L = -\frac{1}{2}\nu_L^T C^T N_L = \frac{1}{2}\nu_L^T C N_L.$$

- $\mathcal{M}$ can be diagonalized by $O^T \mathcal{M} O = \mathcal{M}_d$ with an orthogonal matrix O. You can check that:

$\mathcal{M}_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ with $m_{1,2} = \frac{M}{2} \mp \sqrt{\frac{M^2}{4} + m^2}$, and the mass eigenstates χ_{1L} and χ_{2L} are given by

$$\begin{pmatrix} \chi_{1L} \\ \chi_{2L} \end{pmatrix} = O^T n_L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \nu_L \\ N_L \end{pmatrix}, \text{ with the mixing angle}$$

θ given by $\tan(2\theta) = \frac{2m}{M}$.

Note that χ_1 and χ_2 are Majorana particles.

↳ In the limit $m \ll M$: $m_{\substack{1\\2}} = \frac{M}{2} \mp M\sqrt{\frac{1}{4} + \frac{m^2}{M^2}} \simeq \frac{M}{2} \mp M\left(\frac{1}{2} + \frac{m^2}{M^2}\right)$,
and hence we find $|m_1| \simeq \frac{m^2}{M} \ll |m_2| \simeq M$.

- Now we could repeat our discussion leading to the PMNS-matrix. So the neutrino oscillations will be due not only to flavor mixing, but also to the mixing between ν_L and N_L. For $m \ll M$, however, the mixing angle will be very small.
  - The Majorana mass component M is comparable to the GUT scale and violates lepton number, while the Dirac mass component m is of order of the much smaller EW scale. The smaller eigenvalue m_1 then leads to a very small neutrino mass, comparable to 1eV, which is in qualitative accord with experiments.
- The name „seesaw" mechanism comes from the fact that if $|m_2| \simeq M$ goes up, then $|m_1| \simeq \frac{m^2}{M}$ goes down, and vice versa.

(*) Note that l_{ij} is also momentum-dependent, so for small enough neutrino momenta, l_{ij} becomes small compared to x (distance sun-earth), so one would average over the distance and get a const. probability $\langle P_{\nu_e \to \nu_\mu} \rangle$. For large momenta this is not possible. In fact, for large momenta the high electron density in the sun is crucial for the correct calulation of the neutrino oscillations to match observations, but this lies beyond the scope of this exercise.

② Kaon oscillations and CP violation

(a) The Kaon is the lowest mass particle, which contains an s-quark. Since only the weak interactions don't conserve flavor, the Kaon can only decay via weak interactions.

(b) $P|K^0\rangle = -|K^0\rangle$, $P|\bar{K}^0\rangle = -|\bar{K}^0\rangle$

$C|K^0\rangle = |\bar{K}^0\rangle$, $C|\bar{K}^0\rangle = |K^0\rangle$

$CP|K^0\rangle = -|\bar{K}^0\rangle$, $CP|\bar{K}^0\rangle = -|K^0\rangle$

(c) $|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right)$
$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right)$
with $CP|K_1\rangle = |K_1\rangle$ and $CP|K_2\rangle = -|K_2\rangle$.

(d) The 2-pion states are parity-even, while the 3-pion states are parity-odd (in the ground state).

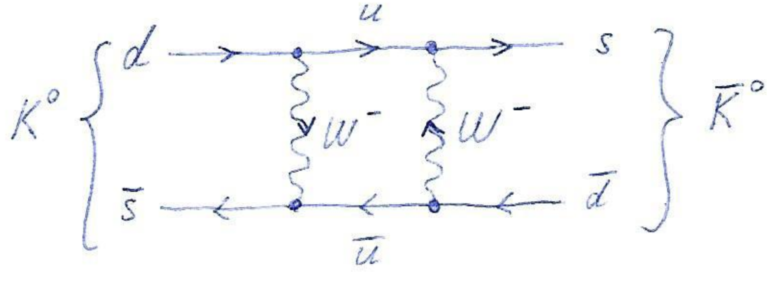
$\Rightarrow CP|\pi\pi\rangle = |\pi\pi\rangle$, $CP|\pi\pi\pi\rangle = -|\pi\pi\pi\rangle$

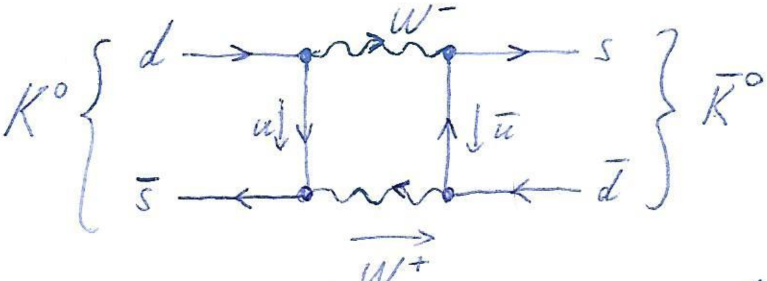
$\hookrightarrow K_1 \to 2\pi$, $K_2 \to 3\pi$

So if we observe $K^0 \to 2\pi$, then actually only the component $|K_1\rangle$ of $|K^0\rangle = \frac{1}{\sqrt{2}}\left(|K_1\rangle + |K_2\rangle\right)$ contributes to the decay.

(e) Now the component $|K_2\rangle$ contributes.

(f)





$\hookrightarrow$ There are also diagrams with c,t instead of u, which are more suppressed due to smaller CKM-factors.

(g) • The diagonal elements of $H_{weak} = \begin{pmatrix} m-\frac{i}{2}\Gamma & -p^2 \\ -q^2 & m-\frac{i}{2}\Gamma \end{pmatrix}$ describe the quantum mechanical (independent) time-evolution of $|K^0\rangle$ and $|\bar{K}^0\rangle$, with the imaginary piece $\sim i\Gamma$ due to decay.

• The off-diagonal elements $\langle K^0|H_{weak}|\bar{K}^0\rangle = -p^2$, $\langle \bar{K}^0|H_{weak}|K^0\rangle = -q^2$ are generated by the above loop diagrams

↳ p and q encode the (non-symmetric) dependence on the CKM-elements.

• The eigenvalues of H_{weak} are $\lambda_\mp = m - \frac{i}{2}\Gamma \mp pq$, with the corresponding eigenvectors $v_\mp = \begin{pmatrix} p \\ \pm q \end{pmatrix}$, so we see that $|K_S^0\rangle$ and $|K_L^0\rangle$ are the eigenstates of H_{weak}.

• We can read off the masses and life-times from $\lambda_\mp$:

K_S^0: $m_S = m - \mathrm{Re}[pq]$, $\tau_S = \left(\Gamma + 2\,\mathrm{Im}[pq]\right)^{-1}$

K_L^0: $m_L = m + \mathrm{Re}[pq]$, $\tau_L = \left(\Gamma - 2\,\mathrm{Im}[pq]\right)^{-1}$

↳ In deriving $\lambda_\mp$, we chose the signs in $m - \frac{i}{2}\Gamma - \lambda = \pm pq$, such that the imaginary part of $\sqrt{p^2q^2}$ is positive, so we know that $\tau_S < \tau_L$. Experiment shows that $m_S < m_L$, so $\mathrm{Re}[pq] > 0$.