(SPECIAL)

W decay: $W^- \to e_L + \bar{\nu}_R$

(a) Precise conventions

$$\boxed{\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1}$$

- helicity of massless left- (right) handed state

$$\psi_R = \begin{pmatrix} 0 \\ u_R \end{pmatrix}, \quad \psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \tag{2}$$

$$m = 0 \quad \Rightarrow \quad \not{p}\psi = 0$$

$$\Rightarrow \begin{pmatrix} 0 & p_0 - \vec{p}\cdot\vec{\sigma} \\ p_0 + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \tag{3}$$

$$\Rightarrow (p_0 - \vec{p}\cdot\vec{\sigma})\, u_R = 0 \quad ; \quad (p_0 + \vec{p}\cdot\vec{\sigma})\, u_L = 0 \tag{4}$$

$$\Downarrow$$

$$h\, u_L = -1/2\, u_L \qquad h\, u_R = 1/2\, u_R \tag{5}$$

$$h \equiv \vec{S}\cdot\hat{p} = \vec{S}\cdot\frac{\vec{p}}{|\vec{p}|} = \frac{\vec{\sigma}}{2}\cdot\frac{\vec{p}}{p^0} \quad (p^0 \equiv |\vec{p}|)$$

$$\Downarrow$$

left-handed fermions: $h = -1/2$ $(\vec{s} \uparrow\downarrow \vec{p})$

right -"- -"- : $h = 1/2$ $(\vec{s} \uparrow\uparrow \vec{p})$

- Polarization vectors of W-boson:

$$\left.\begin{aligned} \varepsilon_{\mu\,T}^{(1)} &= \frac{1}{\sqrt{2}}\ (0;\ +1,\ +i,\ 0)/\sqrt{2} \\ \varepsilon_{\mu\,T}^{(2)} &= \frac{1}{\sqrt{2}}\ (0;\ +1,\ -i,\ 0)/\sqrt{2} \\ \varepsilon_{\mu\,L}^{(3)} &= (0;\ 0,\ 0,\ 1) \end{aligned}\right\} \text{rest-frame} \qquad (6)$$

Notice: $$\sum_i \epsilon_\mu^{(i)} \varepsilon_\nu^{(i)*} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \qquad (7)$$

$(k^0 = m,\ k^i = 0)$

- $$\underbrace{(T_i)_{jk} = -i\,\varepsilon_{ijk}}_{\text{generators of } SU(2) \text{ for the triplet (vector)}} \quad \text{give:} \quad \underbrace{[T_i, T_j] = i\,\varepsilon_{ijk} T_k}_{SO(3) = SU(2)} \qquad (8)$$

$$\Rightarrow \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9)$$

$$\Rightarrow \quad T_3 \, \vec{\varepsilon}_T^{\,(1)} = + \vec{\varepsilon}_T^{\,(1)}$$

$$T_3 \, \vec{\varepsilon}_T^{\,(2)} = - \vec{\varepsilon}_T^{\,(2)} \qquad (10)$$

$$T_3 \, \vec{\varepsilon}_L^{\,(3)} = 0$$

$\Rightarrow \quad \varepsilon_{\mu\,T}^{(1)}$: corresponds to spin +1 along the z-axis ($S_z = 1$)

$\varepsilon_{\mu\,T}^{(2)}$: $S_z = -1$

$\varepsilon_{\mu\,L}^{(3)}$: $S_z = 0$

• Boosting in z-direction:

$$\varepsilon_z{}' = \frac{\varepsilon_z + v\,\varepsilon_0}{\sqrt{1-v^2}}$$

$$\varepsilon_0{}' = \frac{\varepsilon_0 + v\,\varepsilon_z}{\sqrt{1-v^2}} \qquad (11)$$

$$p = \frac{m v}{\sqrt{1-v^2}} \; ; \quad E = \frac{m}{\sqrt{1-v^2}}$$

$$\Rightarrow \quad v = p/E \; ; \quad \frac{1}{\sqrt{1-v^2}} = E/m \qquad (12)$$

$$\Rightarrow \quad \varepsilon'^{(1)}_{\mu\,T} = \varepsilon^{(1)}_{\mu\,T} \quad ; \quad \varepsilon'^{(2)}_{\mu\,T} = \varepsilon^{(2)}_{\mu\,T}$$

– transverse polarization ($\perp \vec{p}$)

$$\varepsilon'^{(3)}_{\mu_L} = \left(\frac{|\vec{p}|}{m} \,;\, 0,\, 0,\, E/m \right)$$ – longitudinal ($\parallel \vec{p}$)

(13)

• Notice : $k_\mu \cdot \epsilon^{\mu(i)} = 0 \quad (\Leftrightarrow \partial_\mu A^\mu = 0)$

normalization : $\varepsilon^{(i)}_\mu \varepsilon^{*(i)\mu} = -1$

Important comment

I took $\epsilon^T = (0; 1, \pm i, 0)/\sqrt{2}$

$$\Leftrightarrow S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Q. What dictates the form of S? why is $T_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ <u>not</u> good?

A. The choice of Pauli matrices.

<u>check</u> $A_i \equiv \bar{\psi}\gamma^i\psi = \psi^\dagger \gamma^0\gamma^i \psi = \psi^\dagger \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \psi$

$$\Rightarrow A_i' = u^\dagger (1 - i\theta_k \sigma_k/2)\, \sigma_i\, (1 + i\theta_k \sigma_k/2)\, u$$

$$= u^\dagger (\sigma_i - i\,\theta_k/2\, [\sigma_k, \sigma_i])\, u =$$

$$= u^\dagger [\sigma_i + \theta_k \epsilon_{kij} \sigma_j] u = u^\dagger \sigma_i u + \epsilon_{ijk}\theta_j u^\dagger \sigma_k u$$

$$\Rightarrow A_i' = A_i - \epsilon_{ijk}\theta_j A_k = A_i + i(\theta_j T_j)_{ik} A_k$$

$$\Rightarrow \boxed{(T_i)_{jk} = -i\,\epsilon_{ijk}}$$

Q.E.D.

Trace formulas

- $$\boxed{\mathrm{Tr}\left[\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta\right] = 4\left(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} + g_{\nu\alpha} g_{\mu\beta}\right)} \qquad (14)$$

- $$\mathrm{Tr}\left[\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_5\right] = c\, \epsilon_{\mu\nu\alpha\beta}$$

use:

$$\gamma_5 = -i\, \gamma_0 \gamma^1 \gamma^2 \gamma^3 = i\, \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

Check:

$$\gamma_0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \begin{pmatrix} -\sigma_2 \sigma_3 & 0 \\ 0 & -\sigma_2 \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_1 \sigma_2 \sigma_3 & 0 \\ 0 & -\sigma_1 \sigma_2 \sigma_3 \end{pmatrix}$$

$$\Rightarrow \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} \checkmark \quad \text{ok}$$

$$\Rightarrow \mathrm{Tr}\left[\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_5\right] = i\, \mathrm{Tr}\left[\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_0 \gamma_1 \gamma_2 \gamma_3\right] = -4i$$

$\Rightarrow \quad c = -4i \quad$ or:

$$\boxed{\mathrm{Tr}\left[\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_5\right] = -4i\, \varepsilon_{\mu\nu\alpha\beta}} \qquad (15)$$

$$\boxed{W - \text{decay}}$$

$$-ig/\sqrt{2}\,\left[W_\mu^- \,\bar{e}_L \gamma^\mu \nu_L + h.c.\right]$$

$$\Downarrow$$

$$|m|^2 = \frac{g^2}{2}\,\epsilon_\mu\, \bar{u}(p)\gamma^\mu L\, v(\ell)\, \bar{v}(\ell)\gamma^\nu L\, u\, \epsilon_\nu^*$$

anti-neutrino (momentum ℓ)

electron of momentum p

$$\Downarrow$$

$$|m|^2 = \frac{g^2}{2}\,\bar{u}(p)\,\not{\epsilon}\, L\, v(\ell)\,\bar{v}(\ell)\,\not{\epsilon}^*\, L\, u$$

next: $\displaystyle\sum_{\text{spins}} |m|^2 = ?$

use: $\displaystyle\sum_s u\bar{u} = \not{p}$

$\displaystyle\sum_s v\bar{v} = \not{\ell}$

$$\Rightarrow \sum_s |m|^2 = g^2/2\ \mathrm{Tr}\left[\not{\epsilon}\,\not{\ell}\,\not{\epsilon}^*\, L\,\not{p}\right] = \frac{g^2}{2}\mathrm{Tr}\left[\not{\epsilon}\,\not{\ell}\,\not{\epsilon}^*\not{p}\, R\right] \qquad (16)$$

$$= \frac{g^2}{2}\mathrm{Tr}\left[\not{\epsilon}\,\not{\ell}\,\not{\epsilon}^*\not{p}\, R\right]$$

$$\Rightarrow \sum_{s} |m|^2 = \frac{g^2}{4} \, \mathrm{Tr}\left[\not{\varepsilon} \, \not{q} \, \not{\varepsilon}^* \, \not{p} \, (1 - \gamma_5) \right] \qquad (17)$$

$$= \frac{g^2}{4} \, \mathrm{Tr}\left[\not{\varepsilon} \, \not{q} \, \not{\varepsilon}^* \, \not{p} \, (1 - \gamma_5) \right]$$

$$\varepsilon = (0;\ +1,\ +i,\ 0)/\sqrt{2} \qquad (S_z^W = +1)$$

$$\varepsilon^* = (0;\ +1,\ -i,\ 0)/\sqrt{2}$$

$$\text{explain} \left\{ \begin{aligned} p_\mu &= (1;\ \sin\theta,\ 0,\ \cos\theta)\,\frac{M_W}{2} \\ q_\mu &= (1;\ -\sin\theta,\ 0,\ -\cos\theta)\,\frac{M_W}{2} \end{aligned} \right. \qquad \text{limit: } (m_e = m_\nu = 0)$$

$$\Downarrow$$

$$\boxed{\text{compute the above expression for } |M|^2 \text{ in (17)}}$$

Differential decay rate

$$\frac{d\Gamma}{d\Omega} = \frac{1}{(2\pi)^2} \int \frac{p^2\,dp}{2p_0} \int \frac{d^3q}{2q_0} \frac{1}{2k_0} \sum_s |M|^2 \delta^{(4)}(k-p-q) \qquad (21)$$

$$k_0 = M_W, \quad \vec{k} = 0 \qquad \begin{pmatrix} p \equiv |\vec{p}\,| ; \\ q \equiv |\vec{q}\,| \end{pmatrix}$$

$$\Downarrow$$

$$\frac{d\Gamma}{d\Omega} = \frac{1}{4\pi^2} \boxed{|M|^2} \frac{1}{2M_W} \int \frac{\cancel{p^2}\,dp}{2\cancel{p_0}} \frac{1}{2\cancel{p_0}} \delta(M_W - 2p)$$

since : $\displaystyle \int \frac{d^3q}{2q_0} \delta^{(3)}(\vec{p}+\vec{q}\,) = \frac{1}{2|\vec{p}\,|} = \frac{1}{2p_0}$

$$\Downarrow$$

$$\boxed{\text{compute } \frac{d\Gamma}{d\Omega} \text{ as a function of } \theta}$$

$\boxed{\text{• use helicity arguments to confirm the } \theta\text{-dependence}}$

Total decay rate

$$\Gamma_W = \int \frac{d\Gamma}{d\Omega}\, d\Omega = 2\pi \int \frac{d\Gamma}{d\Omega} \sin\theta\, d\theta$$

$$\Downarrow$$

this part is trivial and gives

$$\boxed{\Gamma_W(+1) = \frac{g^2 M_W}{48\pi}} \qquad \text{for } S_z^W = +1$$

- compute next $\Gamma_W(-1)$ and compare with $\Gamma_W(+1)$. Are they equal and why?

- Next, do the longitudinal polarization and compare the rates.

Excercise

Compute the decay rate for the ~~averaged~~ averaged spin of W

use:

$$\sum_i \epsilon_\mu^{(i)} \epsilon_\nu^{*(i)} = \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2} \right]$$

and we need:

$$\sum \frac{1}{3} |M|^2 = \frac{g^2}{12} \mathrm{Tr} \Big[-\gamma^\mu \overset{(a)}{\not k} \gamma_\mu \not p (1-\gamma_5) + \not K \not k \overset{(b)}{\not K \not p} (1-\gamma_5) \Big]$$

(c)

$=$

complete the calculation

$\Downarrow$

$$\boxed{\sum \frac{1}{3} |M|^2 = \frac{g^2 M_W^2}{3}}$$

$\Downarrow$

$$\Gamma_W = \frac{1}{4\pi^2}\,\frac{g^2 M_W^2}{3}\int\frac{d^3p}{2p_0}\int\frac{d^3\ell}{2\ell_0}\,\frac{1}{2M_W}\,\delta^{(4)}(p+k-\ell)$$

$$= \frac{g^2 M_W}{48\pi}$$ as expected Why?