Rechenregeln für Poissonklammern

allg. Def.: $\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}$

$c = \text{const.}$ $\{f, c\} = 0$ $\qquad \{f, g\} = -\{g, f\}$

$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$$

$$\{f_1 \cdot f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\}$$

$$\{p_k, g\} = \frac{\partial g}{\partial q_k} \qquad \{q_k, g\} = -\frac{\partial g}{\partial p_k}$$

$$\{p_j, q_k\} = \frac{\partial p_j}{\partial p_i} \frac{\partial q_k}{\partial q_i} - \frac{\partial p_j}{\partial q_i} \frac{\partial q_k}{\partial p_i} = \delta_{jk}$$

$$\{p_j, q_k\} = \delta_{jk} \qquad \{q_j, q_k\} = 0 \qquad \{p_j, p_k\} = 0$$

Jacobi-Identität: (Beweis durch Nachrechnen)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \equiv 0$$

Anwendung: Poisson-Theorem

f und g Bewegungsintegrale $\Rightarrow$ $\{f, g\}$ Bewegungsintegral

(kann neue Bewegungsintegrale liefern)

Bew.: $\frac{d}{dt}\{f, g\} = \frac{\partial}{\partial t}\{f, g\} + \{H, \{f, g\}\} =$

$$= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} - \{f, \{g, H\}\} - \{g, \{H, f\}\} =$$

$$= \left\{\frac{\partial f}{\partial t} + \{H, f\}, g\right\} + \left\{f, \frac{\partial g}{\partial t} + \{H, g\}\right\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}$$

$\Rightarrow$ Beh. $\boxtimes$

Beispiel: Drehimpuls $l_i = \varepsilon_{ilk} r_l p_k$

Algebra: $\{l_i, l_j\} = -\varepsilon_{ijk} l_k$ z.B. $\{l_1, l_2\} = -l_3$

Bew.: $r.S. = -\varepsilon_{ijk}\varepsilon_{kln} r_l p_n = -(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl}) r_l p_n =$
$= -r_i p_j + r_j p_i$

$$l.S. = \{l_i, l_j\} = \varepsilon_{ilk}\varepsilon_{jns}\{r_l p_k, r_n p_s\} =$$

$$= \varepsilon_{ilk}\varepsilon_{jns}\Big(r_l \underbrace{\{p_k, r_n\}}_{\delta_{kn}} p_s + p_k \underbrace{\{r_l, p_s\}}_{-\delta_{ls}} r_n\Big) =$$

$$= -(\delta_{ij}\delta_{ls} - \delta_{is}\delta_{jl}) r_l p_s + (\delta_{ij}\delta_{kn} - \delta_{in}\delta_{jk}) p_k r_n$$

$$= r_j p_i - r_i p_j \quad \blacksquare$$

$$\{\vec{l}^{\,2}, l_j\} = \{l_i l_i, l_j\} = 2 l_i \{l_i, l_j\} = -2\varepsilon_{ijk} l_i l_k = 0$$

$$\{p_i, l_j\} = \varepsilon_{jnk}\{p_i, r_n p_k\} = \varepsilon_{jik} p_k = -\varepsilon_{ijk} p_k$$

Damit: Falls $\underline{l_1, l_2}$ erhalten $\overset{\text{Poisson-Theorem}}{\Longrightarrow}$ $\underline{l_3} = -\{l_1, l_2\}$ erhalten

Falls $\underline{l_3, p_1}$ erhalten $\Rightarrow$ $\{p_1, l_3\} = -\varepsilon_{132} p_2 = \underline{p_2}$ erhalten

Bemerkung: formale Analogie

Klassische Mechanik	Quantenmechanik
p_j, q_k	Operatoren (Matrizen) $\hat{p}_j, \hat{q}_k$
Poissonklammern	Kommutatoren $[A, B] \equiv AB - BA$
$\{p_j, q_k\} = \delta_{jk}$	$\frac{i}{\hbar}[\hat{p}_j, \hat{q}_k] = \delta_{jk}$

↳ Heisenbergsche Unschärferelation

7.3 Kanonische Transformationen

Lagrange-Formalismus: Koordinaten q_i

$\rightarrow$ neue Koord. $Q_i = Q_i(q_k, t)$ $\rightarrow$ ELG invariant

Hamilton-Formalismus:

$$p_i, q_i \longrightarrow P_i(p,q,t),\ Q_i(p,q,t)$$

heißt kanonische Transformation, falls Bewegungsgln. ihre kanonische Form behalten, d.h.

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial H'}{\partial Q_i} \qquad (\square)$$

mit neuer Hamilton-Funktion $H'(P_i, Q_i)$

Variationsprinzip für kanonische Gleichungen

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} (p\dot{q} - H(p,q,t))\, dt =: S[p,q]$$

$$\delta S = \int_{t_1}^{t_2} dt \Big(\delta p\, \dot{q} + \underbrace{p\, \delta\dot{q}}_{\frac{d}{dt}(p\delta q) - \dot{p}\delta q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q \Big) =$$

$$= \int_{t_1}^{t_2} dt \Big(\big(\dot{q} - \frac{\partial H}{\partial p}\big)\delta p - \big(\dot{p} + \frac{\partial H}{\partial q}\big)\delta q \Big) + \underbrace{p \cdot \delta q \Big|_{t_1}^{t_2}}_{=0}$$

$$\delta S \overset{!}{=} 0 \qquad \forall\ \delta p(t),\ \delta q(t)$$

$$\Leftrightarrow \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

damit $\mathcal{P}_i, Q_i$ ($\square$) erfüllen, muss analog gelten

$$\delta \int_{t_1}^{t_2} (\mathcal{P}_i \mathring{Q}_i - H'(\mathcal{P}, Q, t))\, dt = 0 \qquad (a)$$

ebenso wie $$\delta \int_{t_1}^{t_2} (p_i \mathring{q}_i - H(p, q, t))\, dt = 0 \qquad (b)$$

(a), (b) äquivalent, falls Unterschied der Integranden totale Ableitung $\frac{d}{dt} F(q, Q, t)$ ist. $[\delta q(t_{1,2}) = \delta Q(t_{1,2}) = 0]$

$\Rightarrow$ Bedingung für kanonische Transformation

$$\boxed{p_i \mathring{q}_i - H = \mathcal{P}_i \mathring{Q}_i - H' + \frac{dF}{dt}}$$

$\Leftrightarrow \quad dF = p_i\, dq_i - \mathcal{P}_i\, dQ_i + (H' - H)\, dt$

$\Rightarrow \quad p_i = \frac{\partial F}{\partial q_i} \qquad \mathcal{P}_i = -\frac{\partial F}{\partial Q_i} \qquad H' = H + \frac{\partial F}{\partial t}$

$F(q, Q, t)$: <u>Erzeugende der kanonischen Transformation</u>

alternativ: $\Phi(q, \mathcal{P}, t) := F(q, Q, t) + \mathcal{P}_i Q_i$ Legendre-Trafo

$\Rightarrow \quad d\Phi = p_i\, dq_i + Q_i\, d\mathcal{P}_i + (H' - H)\, dt$

$$p_i = \frac{\partial \Phi}{\partial q_i} \qquad Q_i = \frac{\partial \Phi}{\partial \mathcal{P}_i} \qquad H' = H + \frac{\partial \Phi}{\partial t}$$

<u>Beispiele:</u>

1. $F(q, Q, t) = q_k Q_k$

$\Rightarrow \quad p_i = Q_i, \quad \mathcal{P}_i = -q_i, \quad H' = H$

Rollen von p und q vertauscht

<u>p, q</u>: <u>kanonisch konjugierte Variable</u>

2. $\Phi(q, \mathcal{P}_i, t) = f_k(q,t)\,\mathcal{P}_k$

$$\Rightarrow Q_i = \frac{\partial \Phi}{\partial \mathcal{P}_i} = f_i(q,t) \quad \text{reine Koord.trafo}$$

3. Harmonischer Oszillator $\quad H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$

$$F(q,Q) := \frac{m}{2}\omega q^2 \cot Q \qquad \frac{\partial F}{\partial t} = 0 \Rightarrow H' = H$$

$$\Rightarrow p = \frac{\partial F}{\partial q} = m\omega q \cot Q \qquad \mathcal{P} = -\frac{\partial F}{\partial Q} = \frac{m}{2}\omega q^2 \frac{1}{\sin^2 Q}$$

Auflösen: $\quad q = q(\mathcal{P}, Q) = \sqrt{\frac{2\mathcal{P}}{m\omega}} \sin Q$

$$p = p(\mathcal{P}, Q) = \sqrt{2m\omega\mathcal{P}} \cos Q$$

$$\Rightarrow H'(\mathcal{P},Q) = H(p(\mathcal{P},Q), q(\mathcal{P},Q)) = \omega\mathcal{P}\cos^2 Q + \omega\mathcal{P}\sin^2 Q = \underline{\omega\mathcal{P}}$$

Q zyklisch $\Rightarrow \dot{\mathcal{P}} = -\frac{\partial H'}{\partial Q} = 0, \quad \dot{Q} = \frac{\partial H'}{\partial \mathcal{P}} = \omega = \text{const.}$

$$\Rightarrow \mathcal{P} = \frac{H'}{\omega} = \frac{E}{\omega} = \text{const.} \qquad Q = \omega t + \alpha$$

in $\mathcal{P}, Q$: Lineare Bewegung

$\Leftrightarrow$

$$q = \underbrace{\sqrt{\frac{2E}{m\omega^2}}}_{\text{Amplitude}} \sin(\omega t + \alpha) \qquad p = \sqrt{2mE}\cos(\omega t + \alpha) = m\dot{q}$$

in p, q: harmonische Schwingung