7. Kanonischer Formalismus

7.1. Hamilton-Gleichungen (Kanonische Gleichungen)

ELG: $L = L(q_i, \dot{q}_i, t)$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$; $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $i = 1, \dots, N$

Def.: Hamilton-Funktion

$q_i, \dot{q}_i \to q_i, p_i$

$$H = H(p_i, q_i, t) := p_i \dot{q}_i - L$$

Legendre-Transformation

$$dH \equiv \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt =$$

$$= dp_i\, \dot{q}_i + \underline{p_i\, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \underline{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt =$$

$$= \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Kanonische Gleichungen (2N DGL 1. Ordnung)

äquivalent zu ELG (N DGL 2. Ordnung)

es gilt: $\quad \frac{\partial H(p,q,t)}{\partial t} = -\frac{\partial L(q,\dot{q},t)}{\partial t}$

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial p_i}}_{\dot{q}_i} \dot{p}_i + \underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i} \dot{q}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

ohne explizite Zeitabhängigkeit $\quad H(p_i, q_i) = E = \text{const.}$

Energieerhaltung

Beispiel: $L = \frac{m}{2}\dot{q}^2 - U(q)$

ELG: $m\ddot{q} = -\frac{\partial U}{\partial q}$, $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$

$$H(p,q) = p\dot{q} - L = \frac{p^2}{m} - \frac{m}{2}\left(\frac{p}{m}\right)^2 + U(q) = \underline{\frac{p^2}{2m} + U(q)}$$

Kan. Gln.: $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$, $\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial U}{\partial q}$ $\Leftrightarrow$ ELG

7.2. Poisson-Klammern

Betrachte Funktion $f(p_i, q_i, t)$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i =$$

$$= \frac{\partial f}{\partial t} + \underbrace{\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}}_{=:\{H,f\}} \equiv \frac{\partial f}{\partial t} + \{H,f\}$$

Poisson-Klammer

f Bewegungskonstante $\Leftrightarrow \frac{df}{dt} = 0 \Leftrightarrow \frac{\partial f}{\partial t} + \{H,f\} = 0$

falls keine <u>explizite</u> t-Abhängigkeit, d.h. $\frac{\partial f}{\partial t} = 0$:

$$f \text{ erhalten} \Leftrightarrow \{H,f\} = 0$$

Beispiele:

- $H = H(p,q) \Rightarrow \{H,H\} = 0 \rightarrow H$ erhalten
- Impuls p_k erhalten $\Leftrightarrow$

$$0 = \{H, p_k\} = \frac{\partial H}{\partial p_i}\underbrace{\frac{\partial p_k}{\partial q_i}}_{=0} - \frac{\partial H}{\partial q_i}\underbrace{\frac{\partial p_k}{\partial p_i}}_{=\delta_{ik}} = -\frac{\partial H}{\partial q_k}$$

d.h. q_k zyklisch