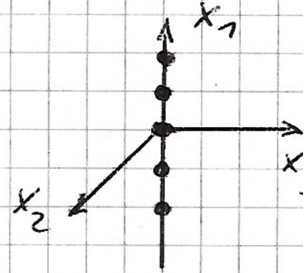
- Spezialfall: Massenverteilung entlang x_3-Achse

$\forall$ Punkte des Körpers: $r_2 = r_3 = 0$ (bez. KS)



$$\Rightarrow J = \sum m \begin{pmatrix} 0 & 0 & 0 \\ 0 & r_1^2 & 0 \\ 0 & 0 & r_1^2 \end{pmatrix}$$ starrer Rotator

$$\Rightarrow \quad J_2 = J_3 = \sum m r_1^2 \ , \quad J_1 = 0$$

nur 2 Freiheitsgrade der Drehung (um x_2, x_3)

- <u>Satz von Steiner</u>

Sei: J_{ij} definiert mit Ursprung im Schwerpkt., $\sum m r_i = 0$

Betrachte $J'_{ij} = \sum m (r_k'^2 \delta_{ij} - r_i' r_j')$

wobei $r_i' = r_i - a_i$ konstante Verschiebung d. Ursprungs

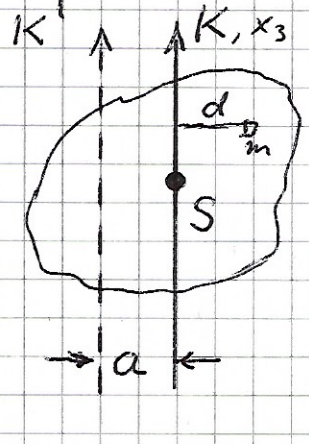
$$\Rightarrow \qquad \boxed{J'_{ij} = J_{ij} + M (a_k^2 \delta_{ij} - a_i a_j)} \qquad (*)$$

<u>Beweis:</u> $\sum m r_i' r_j' = \sum m (r_i - a_i)(r_j - a_j) =$

$$= \sum m r_i r_j - \underbrace{(\sum m r_i)}_{=0} a_j - \underbrace{(\sum m r_j)}_{=0} a_i + \sum m a_i a_j$$

$$= \sum m r_i r_j + M a_i a_j \ , \quad \Rightarrow \quad \sum m r_k'^2 = \sum m r_k^2 + M a_k^2 \ \Rightarrow (*)$$

- Trägheitsmoment bez. beliebiger Achse K'



$$J_K := \sum m d^2 = \sum m (r_1^2 + r_2^2) = J_{33}$$

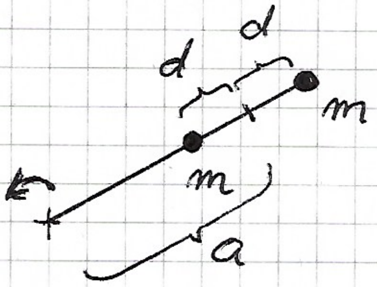
Satz von Steiner $\Rightarrow$

$$\underline{J'_{33}} = J_{33} + M (a_1^2 + a_2^2) = \underline{J_{33} + M a^2}$$

bez. Achse $K' \parallel K$ m. Abstand a, bez. K durch Schwerpkt. S

Beispiel 1

Rotator $J = J_3 = \sum m r_1^2$



i.) direkt:

$$J = m(a-d)^2 + m(a+d)^2 = 2m(d^2+a^2)$$

$\overset{\wedge}{=}$ ✓

ii.) Steiner: $J = \underset{\substack{\uparrow \\ J \text{ um Schwerpunkt}}}{J_S} + Ma^2 = 2md^2 + 2ma^2$

Beispiel 2

homogene Kugel, Masse M, Radius a

$$J_1 = J_2 = J_3 = J = \underline{\underline{\frac{2}{5} M a^2}}$$

$\rho = \dfrac{3M}{4\pi a^3}$ $\qquad J_{ij} = \int d^3r\, \rho \left(r_k^2 \delta_{ij} - r_i r_j \right)$

$$J_{12} = -\rho \int_K d^3r\, r_1 r_2 \propto \int_0^{2\pi} \underbrace{\sin\varphi \cos\varphi}_{\frac{1}{2}\sin 2\varphi}\, d\varphi = 0 = J_{13} = J_{23}$$

$\Rightarrow$ J diagonal, $\quad J_{11} = J_1, \quad J_{22} = J_2, \quad J_{33} = J_3$

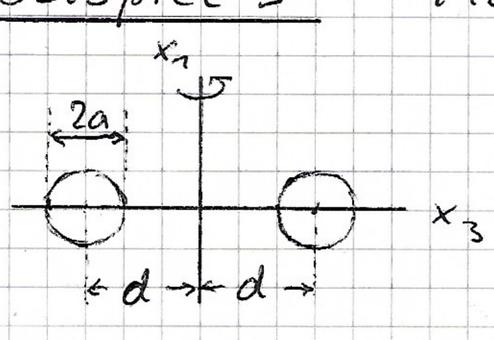
$$J_1 = J_2 = J_3 = \rho \int_K d^3r \left(r_2^2 + r_3^2 \right) = 2\rho \int_K d^3r\, r_3^2 =$$

$$= 2\rho \int_K 2\pi \sin\vartheta\, d\vartheta\, r^2 dr \cdot (r\cos\vartheta)^2 =$$

$$= 4\pi\rho \int_{-1}^{1} d\cos\vartheta \cdot \cos^2\vartheta \int_0^a dr\, r^4 = 4\pi\rho \cdot \frac{2}{3} \cdot \frac{a^5}{5} = \underline{\frac{2}{5} M a^2}$$

Beispiel 3

Hantel (2 homogene Kugeln, jew. Masse M)



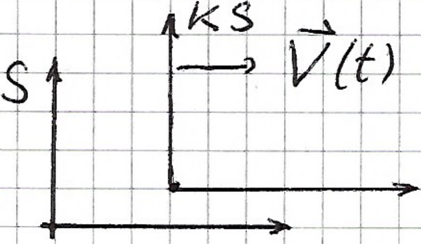
$$J_1 = 2 \cdot \left[\frac{2}{5} M a^2 + M d^2 \right] = J_2$$

$$J_3 = \frac{4}{5} M a^2$$

6.2. Bewegung im beschleunigten Bezugssystem

bisher: Bewegung bezogen auf Inertialsystem (IS)

jetzt: Betrachte Massenpunkt in Nicht-Inertialsystem (z.B. KS)



a.) Translation

$$\underset{\text{in IS}}{v_{0i}} = \underset{\text{in KS}}{v_i} + V_i(t)$$

↑ vorgegebene Fkt.

$$L = \frac{m}{2} v_{0i}^2 - U = \frac{m}{2} v_i^2 + \underbrace{m v_i V_i}_{-m r_i \frac{d}{dt} V_i + \frac{d}{dt}(m r_i V_i)} + \frac{m}{2} V_i^2 - U$$

irrelevant

$$= \frac{m}{2} v_i^2 \underbrace{- m W_i(t)}_{\text{eff. Kraftfeld}} r_i - U \qquad W_i = \frac{d}{dt} V_i$$

Beschleunigung von KS

b.) Rotation $\qquad v_{0i} = v_i + \varepsilon_{ijk}\, \omega_j(t)\, r_k$

↑ vorgegeben

$\Rightarrow$

$$L = \frac{m}{2} v_{0i}^2 - U = \frac{m}{2} v_i^2 + m v_i \varepsilon_{ijk} \omega_j r_k + \frac{m}{2} (\varepsilon_{ijk} \omega_j r_k)^2 - U$$

$$\frac{\partial L}{\partial v_l} = m v_l + m \varepsilon_{ljk} \omega_j r_k \qquad \frac{\partial v_i}{\partial v_l} = \delta_{il}$$

$$\frac{\partial L}{\partial r_l} = m v_i \varepsilon_{ijl} \omega_j + m(\varepsilon_{ijk} \omega_j r_k) \varepsilon_{inl} \omega_n - \frac{\partial U}{\partial r_l}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_l} = \frac{\partial L}{\partial r_l} \quad (ELG) \quad \Rightarrow$$

$$m\dot{v}_l = -m\,\varepsilon_{ljk}\,\dot{\omega}_j\,r_k - m\,\varepsilon_{ljk}\,\omega_j\,v_k + m\,v_k\,\varepsilon_{kjl}\,\omega_j +$$
$$+ m\,\varepsilon_{lin}\left(\varepsilon_{ijk}\,\omega_j\,r_k\right)\omega_n - \frac{\partial U}{\partial r_l} =$$
$$= m\,\varepsilon_{lkj}\,r_k\,\dot{\omega}_j + 2m\,\varepsilon_{lkj}\,v_k\,\omega_j + m\,\varepsilon_{lin}\left(\varepsilon_{ijk}\,\omega_j\,r_k\right)\omega_n - \frac{\partial U}{\partial r_l}$$

$\Leftrightarrow$

$$m\dot{\vec{v}} = m\,\vec{r}\times\dot{\vec{\omega}} + \underbrace{2m\,\vec{v}\times\vec{\omega}}_{\text{Coriolis-}} + \underbrace{m(\vec{\omega}\times\vec{r})\times\vec{\omega}}_{\text{Zentrifugalkraft}} - \frac{\partial U}{\partial \vec{r}}$$

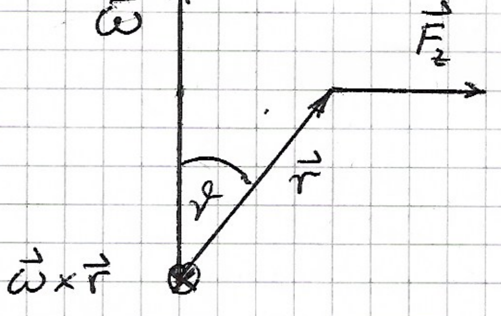
gleichförmige Rotation : $\dot{\vec{\omega}} = 0$

Impuls $p_l = \frac{\partial L}{\partial v_l} = m\left(v_l + \varepsilon_{ljk}\,\omega_j\,r_k\right) = m\,v_{0l} = p_{0l}$

Energie $E = v_i \frac{\partial L}{\partial v_i} - L = \frac{m}{2}v_i^2 \underbrace{- \frac{m}{2}\left(\varepsilon_{ijk}\,\omega_j\,r_k\right)^2}_{\text{Zentrifugalenergie}} + U$

Corioliskraft : $\perp \vec{v}$, $\perp \vec{\omega}$

Zentrifugalkraft : $\vec{F}_z = m(\vec{\omega}\times\vec{r})\times\vec{\omega}$



$|\vec{\omega}\times\vec{r}| = \omega\,r\sin\vartheta$

$|\vec{F}_z| = m\,\omega^2\,r\sin\vartheta = m\,\omega^2\,r_\perp$