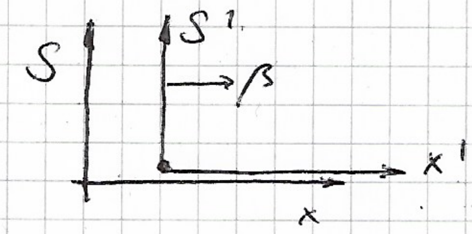
Relativistische Mechanik

1. Lorentz-Trafo aus Invarianz von $t^2 - x^2 = 0$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \qquad \begin{aligned} t' &= At + Bx \\ x' &= Ct + Dx \end{aligned}$$



a.) Zeigen Sie, dass $\frac{C}{D} = -\beta$ (1)

$$x' = 0 \iff \frac{C}{D} = -\frac{x}{t} = -\beta$$

b.) Mit $[t^2 - x^2 = 0 \Rightarrow t'^2 - x'^2 = 0]$ folgt

$$A + B = C + D \qquad (2)$$
$$A - B = D - C \qquad (3)$$

$$t'^2 - x'^2 = (At + Bx)^2 - (Ct + Dx)^2 \underset{x = \pm t}{=} [(A \pm B)^2 - (C \pm D)^2] t^2 \stackrel{!}{=} 0 \quad \forall t$$

$$\Rightarrow (A \pm B)^2 = (C \pm D)^2 \Rightarrow \left\{ \begin{aligned} A + B &= \pm (C + D) \\ A - B &= \pm (C - D) \end{aligned} \right\} \underset{\text{für } \beta = 0:\ B = C = 0,\ A = D = 1}{\Longrightarrow} \begin{aligned} A + B &= C + D \\ A - B &= D - C \end{aligned}$$

c.) Zeige $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = A(\beta^2) \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$

$$(2), (3) \Rightarrow A = D, \quad B = C = -\beta A, \quad A = A(\beta^2)$$

d.) Bestimme A

$$\Lambda^{-1} = A \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \Rightarrow \Lambda^{-1} \Lambda = A^2 \begin{pmatrix} 1 - \beta^2 & 0 \\ 0 & 1 - \beta^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A = \frac{1}{\sqrt{1 - \beta^2}} \equiv \gamma, \quad \Lambda = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \quad \square$$

e.) Galilei-Trafo

$$\Lambda = \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix} \overset{a.)}{=} \begin{pmatrix} 1 & 0 \\ -\beta D & D \end{pmatrix} \qquad D = ?$$

$$\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ \beta D & D \end{pmatrix}, \quad \Lambda^{-1}\Lambda = \begin{pmatrix} 1 & 0 \\ \beta(D-D^2) & D^2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow D^2 = 1, \quad D - D^2 = 0 \quad \Rightarrow D = +1, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \qquad \begin{matrix} t' = t \\ x' = x - \beta t \end{matrix}$$

2.a.) Zeige $\Lambda(\eta_1)\,\Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2)$

$$\begin{pmatrix} \mathrm{ch}\,\eta_1 & -\mathrm{sh}\,\eta_1 \\ -\mathrm{sh}\,\eta_1 & \mathrm{ch}\,\eta_1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{ch}\,\eta_2 & -\mathrm{sh}\,\eta_2 \\ -\mathrm{sh}\,\eta_2 & \mathrm{ch}\,\eta_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{ch}\,\eta_1\,\mathrm{ch}\,\eta_2 + \mathrm{sh}\,\eta_1\,\mathrm{sh}\,\eta_2 & -(\mathrm{ch}\,\eta_1\,\mathrm{sh}\,\eta_2 + \mathrm{sh}\,\eta_1\,\mathrm{ch}\,\eta_2) \\ \dots & \dots \end{pmatrix} = (*)$$

$$\mathrm{ch}\,x = \tfrac{1}{2}(e^{x} + e^{-x}) \qquad \mathrm{sh}\,x = \tfrac{1}{2}(e^{x} - e^{-x})$$

$$\mathrm{ch}\,\eta_1\,\mathrm{ch}\,\eta_2 = \tfrac{1}{4}(e^{\eta_1+\eta_2} + e^{-\eta_1-\eta_2} + e^{\eta_1-\eta_2} + e^{\eta_2-\eta_1})$$

$$\mathrm{sh}\,\eta_1\,\mathrm{sh}\,\eta_2 = \tfrac{1}{4}(e^{\eta_1+\eta_2} + e^{-\eta_1-\eta_2} - e^{\eta_1-\eta_2} - e^{\eta_2-\eta_1}) \qquad \oplus = \mathrm{ch}(\eta_1+\eta_2)$$

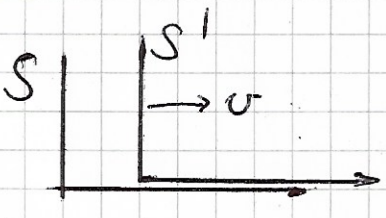
$$\mathrm{ch}\,\eta_1\,\mathrm{sh}\,\eta_2 = \tfrac{1}{4}(e^{\eta_1+\eta_2} - e^{-\eta_1-\eta_2} - e^{\eta_1-\eta_2} + e^{\eta_2-\eta_1}) \xrightarrow{+\{1\leftrightarrow 2\}} \mathrm{sh}(\eta_1+\eta_2)$$

$$\Rightarrow (*) = \begin{pmatrix} \mathrm{ch}(\eta_1+\eta_2) & -\mathrm{sh}(\eta_1+\eta_2) \\ -\mathrm{sh}(\eta_1+\eta_2) & \mathrm{ch}(\eta_1+\eta_2) \end{pmatrix} = \Lambda(\eta_1+\eta_2)$$

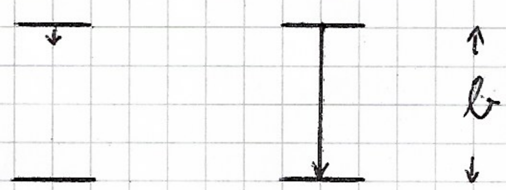
2 b.) Zeige $v = th(\eta_1+\eta_2) = \dfrac{th\eta_1 + th\eta_2}{1 + th\eta_1\, th\eta_2} = \dfrac{v_1+v_2}{1+v_1 v_2}$

$$th(\eta_1+\eta_2) = \frac{sh(\eta_1+\eta_2)}{ch(\eta_1+\eta_2)} = \frac{ch\eta_1\, sh\eta_2 + sh\eta_1\, ch\eta_2}{ch\eta_1\, ch\eta_2 + sh\eta_1\, sh\eta_2} = \frac{th\eta_1 + th\eta_2}{1 + th\eta_1\, th\eta_2}$$

3. Zeitdilatation

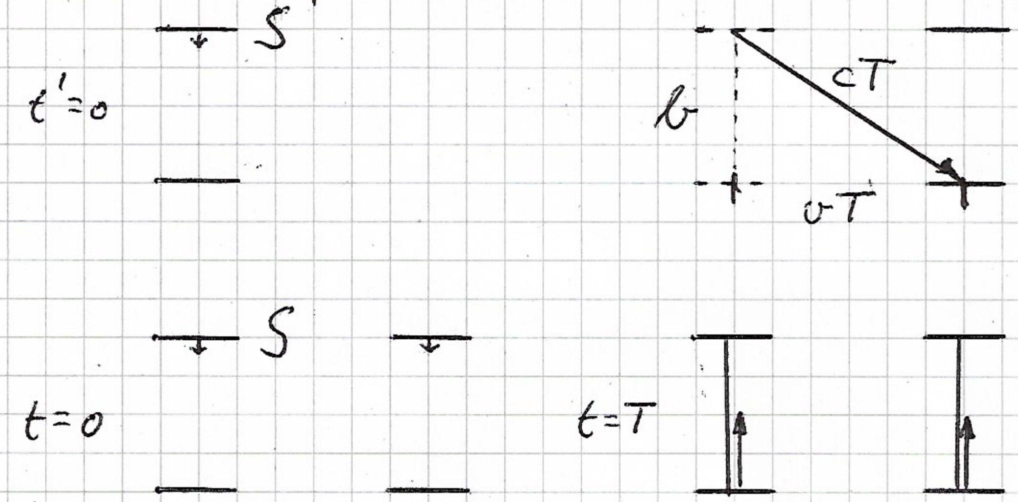


Uhr in S'



$t'=0$ $\quad t'=T' = b/c$

in S:



$$c^2T^2 = v^2T^2 + b^2$$

$$T^2 = \frac{b^2}{c^2-v^2} = \frac{T'^2}{1-\beta^2}$$

$$\Leftrightarrow \quad T' = \sqrt{1-\beta^2}\; T$$