"QCD and Standard Model"
Problem Set 10

Chiral symmetry breaking and sigma models

At the classical level the Lagrangian of QCD with two massless quarks (u and d) is invariant under the global symmetry group

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_A \otimes U(1)_V \tag{1}$$

In this exercise we focus on the global chiral symmetries $SU(2)_L$ and $SU(2)_R$.

a) Write down the QCD Lagrangian for two massless quarks and identify the global chiral symmetries $SU(2)_L$ and $SU(2)_R$ under which it is invariant. How would this symmetry group generalize if we add more massless quarks to the theory?

b) Argue that a condensate

$$< \bar{u}u > = < \bar{d}d > = v^3 \tag{2}$$

breaks $SU(2)_L \otimes SU(2)_R$ down to its diagonal subgroup $SU(2)_V$. How many Goldstone bosons emerge? How many Goldstone bosons emerge in the case of three massless quarks when

$$SU(3)_L \otimes SU(3)_R \longrightarrow SU(3)_V\,? \tag{3}$$

c) In particle physics, we often use phenomenological so-called "sigma model Lagrangians" in order to describe the dynamics of the (pseudo-)Goldstone bosons at low energies. We shall now investigate Gell-Mann and Levy linear and non-linear sigma models which correspond to the QCD Lagrangian with two massless quarks.

Consider the Lagrangian density of the linear sigma model with the four scalar fields $\phi^{i\bar{j}}$ transforming linearly under $SU(2)_L \otimes SU(2)_R$,

$$\mathcal{L} = \frac{1}{2}|\partial_\mu \phi|^2 + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \tag{4}$$

where $\lambda > 0$ and barred indices belong to $SU(2)_R$. Minimize the potential, take the VEV v of the scalar fields in such a way that the symmetry is broken into the diagonal subrgroup and identify the (pseudo-)Goldstone bosons π^1, π^2 and π^3. Expand the Lagrangian around the VEV by introducing a fluctuation, $\sigma(x)$, and write it down in terms of σ and π^a $(a = 1, 2, 3)$.

d) Argue what happens physically to σ in the double scaling limit $\mu^2 \to \infty$, $\lambda \to \infty$, v^2 fixed. Argue that the fields satisfy the constraint $\pi^a\pi^a + \sigma^2 = -2v\sigma$ in this limit.

e) Plugging this constraint into the Lagrangian of the linear sigma model leads to the so-called non-linear sigma model for the three Goldstone bosons π^a

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2}\frac{\pi^a \partial_\mu \pi^a \pi^b \partial^\mu \pi^b}{v^2 - \pi^a \pi^a} \tag{5}$$

Expand this Lagrangian and show that it can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{6v^2}\left((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a \partial_\mu \pi^b \partial^\mu \pi^b\right) + \mathcal{O}(\pi^6) \tag{6}$$

f) In the so-called exponential representation with an SU(2) field U

$$U = \exp\left\{ i\frac{\pi^a \sigma^a}{f_\pi} \right\} \,, \tag{7}$$

with $f_\pi \approx v$, we can construct the effective Lagrangian for the non-linear sigma model

$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U \right) \,, \tag{8}$$

which is also called chiral Lagrangian. Convince yourself that this object is invariant under SU(2). Under what representation of SU(2) do the fields π^a transform?

g) Argue that we can organise the chiral Lagrangian in terms of the number of derivatives of the exponential $U(x)$. Why there are no terms containing only $U(x)$ and without derivatives?

h) Show that a pion mass term can be introduced in this language by adding

$$\delta\mathcal{L}_{\mathrm{mass}} = v^3 \mathrm{Tr}\left(MU + M^\dagger U^\dagger \right) \tag{9}$$

where

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2} \,. \tag{10}$$

Argue that we cannot remove the phase by making an $SU(2)_L \times SU(2)_R$ transformation. However, we can trust the experimental value $|\theta| < 10^{-9}$ and set it to zero. Guess how M should transform in such a way that $\delta\mathcal{L}_{\mathrm{mass}}$ is invariant under $SU(2)_L \times SU(2)_R$.