Prof. Dr. Stefan Hofmann Sommersemester 2020

Übungen zu Theoretischer Mechanik (T1)

Blatt 13

1 Hamilton Im Magnetfeld

Im Folgenden betrachten wir ein geladenes Teilchen der Ladung e und Masse m, welches sich in einem homogenen Magnetfeld $\boldsymbol{B}$ und elektrostatischem Potential ϕ bewege. Überzeugen Sie sich, dass die Hamiltonfunktion als Summe von potenzieller und kinetischer Energie die Form

$$H = \frac{1}{2m}[\boldsymbol{p} - \frac{1}{2}e(\boldsymbol{B} \times \boldsymbol{r})]^2 - e\phi \tag{1}$$

hat.

(i) Nutzen Sie die Hamiltongleichung um zu zeigen, dass $d/dt(m\dot{\boldsymbol{r}})$ gleich der Lorentzkraft ist.

First of all, we rewrite the Hamtilonian in terms of indices notation,

$$H = \frac{1}{2m}\left[p_k - \frac{e}{2}\epsilon_{ijk}B_i q_j\right]^2 - e\phi(\boldsymbol{q}).$$

In general we can also assume $\phi(\boldsymbol{q})$ has spatial dependence which is not homogeneous. Recall the Hamilton's equation:

$$\frac{dp_m}{dt} = -\frac{\partial H}{\partial q_m}, \qquad \frac{dq_m}{dt} = \frac{\partial H}{\partial p_m}$$

Plug the Hamiltonian H into the Hamilton's equation on the left,

$$\begin{aligned}\frac{dp_m}{dt} &= -\frac{1}{m}\left[p_k - \frac{e}{2}\epsilon_{ijk}B_i q_j\right]\frac{\partial}{\partial q_m}\left[p_k - \frac{e}{2}\epsilon_{ijk}B_i q_j\right] - \frac{\partial}{\partial q_m}(e\phi)\\ &= \frac{e}{2m}\left[p_k - \frac{e}{2}\epsilon_{ijk}B_i q_j\right]\epsilon_{ijk}B_i\delta_{jm} - e\frac{\partial\phi}{\partial q_m}.\end{aligned}$$

Similarly, for the Hamtilon's equation on the right,

$$\frac{dq_m}{dt} = \frac{1}{m}\left[p_k - \frac{e}{2}\epsilon_{ijk}B_i q_j\right]\delta_{mk} = \frac{1}{m}\left[p_m - \frac{e}{2}\epsilon_{ijm}B_i q_j\right]$$

and the corresponding time derivative gives another equation describing $\dot{p}_m$,

$$\dot{p}_m = m\ddot{q}_m + \frac{e}{2}\epsilon_{ijm}B_i\dot{q}_j$$

On the other hand, if we insert dq_m/dt into dp_m/dt,

$$\dot{p}_m = e\left[\frac{1}{2}\epsilon_{ijk}B_i\delta_{jm}\dot{q}_k - \frac{\partial\phi}{\partial q_m}\right] = e\left[\frac{1}{2}\epsilon_{kim}\dot{q}_k(t)B_i - \frac{\partial\phi}{\partial q_m}\right],$$

Now, if we take the above two equation into consideration by eliminating $\dot{p}_m$,

$$m\ddot{q}_m + \frac{e}{2}\epsilon_{ijm}B_i\dot{q}_j = \frac{e}{2}\epsilon_{kim}\dot{q}_k(t)B_i - e\frac{\partial\phi}{\partial q_m},$$

Recall that the electric field is related to the gradient of electric potential ϕ by $E_m = -\partial\phi/\partial q_m$. Next, switching the indices on the left hand side of the equation, $\epsilon_{ijm} = -\epsilon_{jim}$, we obtain,

$$m\ddot{q}_m = e\big[E_m + \epsilon_{ijm}\dot{q}_i(t)B_j\big]$$

which is simply the Lorentz force,

$$m\ddot{\boldsymbol{q}}(t) = e\left[\boldsymbol{E} + \dot{\boldsymbol{q}}(t) \times \boldsymbol{B}\right]$$

(ii) Unter welcher Bedingung ist die Impulskomponente entlang $\boldsymbol{B}$ erhalten?

Recall dp_k/dt from the Hamtilon's equation,

$$\frac{dp_k}{dt} = \frac{e}{2}\epsilon_{ijk}\dot{q}_i(t)B_j - e\frac{\partial\phi}{\partial q_m}$$

In order that the momentum along B being conserved, we first require,

$$\frac{\partial\phi}{\partial q_m} = 0$$

This means that the electric potential has to be homogeneous ($\boldsymbol{E} = 0$) in the physical system. Next, given $\partial\phi/\partial q_m = 0$,

$$\frac{dp_k}{dt} = \frac{e}{2}\epsilon_{ijk}\dot{q}_i(t)B_j$$

Notice that the Levi-Civita symbol is zero when its indices are repeated, $\epsilon_{iik} = 0$. The rate of change of momentum dp_k/dt that is parallel/antiparallel to B_j is zero because,

$$\frac{dp_j}{dt} = \frac{e}{2}\epsilon_{ijj}\dot{q}_i(t)B_j = 0$$

In this case, $dp_j/dt = 0$ means that the momentum p_j is conserved.

Based on our observation on the property of the Levi-Civita symbol, the physical force is non-zero only along the direction which is perpendicular to the magnetic field and the particle motion. For example, if we consider a simple case where,

$$\dot{\boldsymbol{q}}(t) = \begin{pmatrix} 0 \\ \dot{\theta}(t) \\ 0 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix},$$

Then the for $dp(t)/dt$,

$$\frac{dp_r}{dt} = \frac{e}{2}\epsilon_{\theta zr}\dot{\theta}(t)B_z = \frac{er\omega(t)B}{2}, \qquad \frac{dp_\theta}{dt} = \frac{e}{2}\epsilon_{\theta z\theta}\theta(t)B_z = 0, \qquad \frac{dp_z}{dt} = \frac{e}{2}\epsilon_{\theta zz}\theta(t)B_z = 0$$

where $\epsilon_{\theta z\theta} = \epsilon_{\theta zz} = 0$. In this sceanario, the angular momentum p_θ and the linear momentum p_z is conserved.

(iii) Welche Erhaltungsgröße folgt aus der Zeitunabhängigkeit der Hamiltonfunktion?

From the last exercise sheet, we know that the Hamiltonian function $H\big(r, p_r, \theta, p_\theta, z, p_z\big)$ of a physical system can be constructed by replacing the generalized velocity in the total energy $E\big(r, \dot{r}(p_r), \theta, \dot{\theta}(p_\theta, r), z, \dot{z}(p_z)\big)$,

$$H\big(r, p_r, \theta, p_\theta, z, p_z\big) = E\big(r, \dot{r}(p_r), \theta, \dot{\theta}(p_\theta, r), z, \dot{z}(p_z)\big)$$

If the Hamiltonian H is time independent,

$$\frac{dH}{dt} = \frac{dE}{dt} = 0$$

then the total energy E of the physical system is also time independent. This implies that the energy E is a conserved quantity if the Hamiltonian function is time independent.

2 Oszillierende Wand

Betrachten Sie ein Teilchen der Masse m, welches über eine horizontale Feder der Federkonstante k und Ruhelänge l an einer Wand befestigt ist. Die Wand bewege sich dabei mit $X_{\text{Wand}} = A\cos(\omega t)$ hin und her. Bestimmen Sie die Hamiltonfunktion des Systems in Abhängigkeit von der Auslenkung der Feder aus der Ruhelage. Stellen Sie die Hamiltongleichungen auf und bestimmen Sie welche Erhaltungsgrößen das System hat.

First of all, the total energy of the oscillating mass m_1 is,

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}k(x_1 - x_2)^2$$

where we define,

$$X_{\text{WAND}} := x_2 = a\cos(\omega t)$$

Then we write the Hamiltonian by replacing the velocity coordinate of the energy

$$H = \frac{p_1^2}{2m} + \frac{1}{2}k(x_1 - x_2)^2$$

where the linear momentum is given by $p_1 = m_1\dot{x}_1$. If we evaluate the Hamtilon's equation,

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1} = -k(x_1 - x_2), \qquad \frac{dx_1}{dt} = \frac{p}{m}$$

Since $dp_1/dt \neq 0$, this shows that the linear momentum of the oscillating mass is not conserved. However, this is the consequence where the kinetic energy of the oscillating wall is not included in the physical system. For example, if we include the motion of the wall into the total energy of the system,

$$H_{12} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k(x_1 - x_2)^2$$

Then, from the Hamilton's equation, we have,

$$\frac{dp_1}{dt} = -k(x_1 - x_2), \qquad \frac{dp_2}{dt} = k(x_1 - x_2) = -\frac{dp_1}{dt}$$

This implies that although the linear momentum of each individual mass is not conserved, the total amount of linear momentum of this physical system is still conserved. The same argument holds for the total energy of the system. In the absence of external force,

$$\frac{dH_{12}}{dt} = 0$$

However, for a physical system that excluded the momentum of the wall,

$$H = H_{12} - \frac{p_2^2}{2m}$$

such that,

$$\frac{dH}{dt} = \frac{dH_{12}}{dt} - \frac{d}{dt}\left(\frac{p_2^2}{2m}\right) = \frac{d}{dt}\left(\frac{p_2^2}{2m}\right) \neq 0$$

which shows that the total energy E is not conserved if the kinetic energy of m_2 is not included.