Arnold Sommerfeld Center | Ludwig–Maximilians–Universität München

Prof. Dr. Stefan Hofmann | Sommersemester 2020

Übungen zu Theoretischer Mechanik (T1)

Blatt 8

1 Picard-Iteration

Betrachten Sie erneut das Anfangswertproblem

$$\dot{x}(t) = t^2 + x(t)^2, \quad x(0) = 0 \tag{1}$$

welches eine eindeutige Lösung auf $[0, \frac{1}{\sqrt{2}})$ besitzt. Bestimmen Sie die ersten vier Funtionen $x_0(t), \cdots, x_3(t)$ der Picard-Iteration und plotten Sie diese gegen die tatsächliche Lösung.
Hinweis: Für die exakte Lösung verwenden Sie am besten einen Computer.
For the initial value problem $f(x,t) = \dot{x}(t)$. Starting with the initial condition $x(0) = 0$, we iterature according to the recurrent formula for Picard iteration:

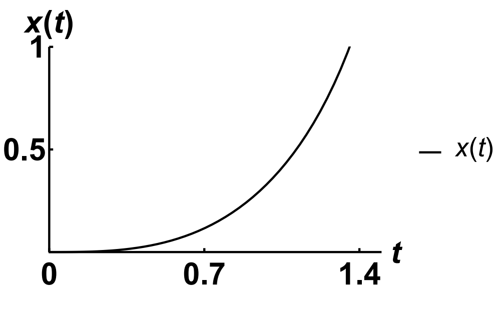
$$x_{n+1}(t) = x_0 + \int_{x_0}^{x(t)} f_n(t,x)dt, \qquad n = 0,1,2,3,...$$

Then,

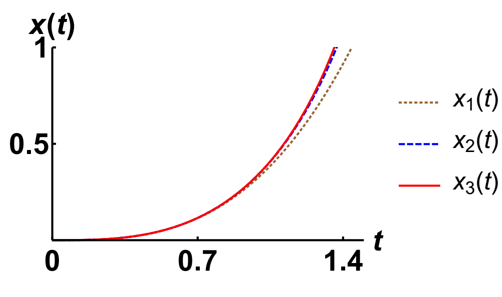
$$x_1(t) = \int_{x_0}^{x(t)} t^2 \, dt = \frac{t^3}{3}$$

$$x_2(t) = \int_{x_0}^{x(t)} \left[t^2 + \frac{t^6}{3}\right] dt = \frac{t^3}{3} + \frac{t^7}{63}$$

$$x_3(t) = \int_{x_0}^{x(t)} \left[t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2\right] dt = \frac{t^3}{3} + \frac{t^7}{63} + \frac{2t^{11}}{2079} + \frac{t^{15}}{59535}$$



(a) Exact solution of $x(t)$



(b) Picard iterated solution of $x_{1,2,3}(t)$

2 Methode der sukzessiven Approximation

In der Vorlesung haben Sie die *Methode der sukzessiven Approximation* kennengerlernt, mit deren Hilfe sich approximative Lösungen von Differentialgleichungen finden lassen. In dieser Aufgabe wollen wir diese Methode dazu verwenden, approximative Lösungen der sogenannten *Langevin-Gleichung* zu finden, die bspw.

dazu benutzt werden kann, die Bewegung eines Objekts in einer Flüssigkeit zu beschreiben[1]. Im Sinne der Einfachheit wollen wir uns hierbei auf eine Raumdimension beschränken. Die Langevin-Gleichung lautet

$$m\ddot{q}(t) = -\lambda\dot{q}(t) + \eta f(t, q(t)). \tag{2}$$

Die Kraft, die bspw. durch die chaotische Bewegung der Flüssigkeitsmoleküle auf das Objekt wirkt, ist durch die einheitenlose Funktion $f(t, x)$ beschrieben, während λ und η zwei Konstanten bezeichnen.

(i) Welche Interpretation haben die Konstanten λ und η? Geben Sie deren Einheiten an.

λ is the damping coefficient with the units of $[m/t]$; while η has the unit of force because f is unitless, it is known as the noise factor.

Um eine konkrete Lösung zu ermöglichen, betrachten wir nun als einfaches Beispiel die Funktion $f(t, x) = \cos(\omega t - kx)$, wobei ω und k zwei weitere, unbekannte Parameter seien. Der Zustand des Objektes von Interesse sei im Folgenden beschrieben durch seine Kurve im Phasenraum, $\alpha(t) = (q(t), v(t))^T$.

(ii) Überführen Sie die Langevin-Gleichung in ein System von Differentialgleichungen erster Ordnung, d.h. bringen Sie sie in die Form $\dot{\alpha}(t) = (\dot{q}(t), \dot{v}(t))^T = \ldots$, wobei $v(t) = \dot{q}(t)$.

The system of first order differential equations describing the Langevin equation is given by $\dot{q}(t)$ and $\dot{v}(t)$ as follow:

$$\dot{q}(t) = v(t)$$
$$\dot{v}(t) = -\frac{\lambda}{m}v(t) + \frac{\eta}{m}\cos(\omega t - kq(t))$$

In matrix notation,

$$\begin{pmatrix}\dot{q}(t)\\ \dot{v}(t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ \eta/m & -\lambda/m\end{pmatrix}\begin{pmatrix}\cos(\omega t - kq(t))\\ v(t)\end{pmatrix}$$

Nehmen Sie nun an, der Zustand des Objekts zum Zeitpunkt $t = 0$ sei gegeben durch $\alpha_0 = \alpha(0) = (0, 0)^T$.

(iii) Nutzen Sie die Methode der sukzessiven Approximation, um $\alpha_n(t)$ bis einschließlich $n = 3$ zu bestimmen. $\alpha_n(t)$ bezeichne hier, in der Konvention der Vorlesung, das n-te Glied der Reihe $\{\alpha_n\}_n$, die durch die Rekursionsrelation

$$\alpha_{n+1}(t) = \alpha_0 + \int_0^t \mathrm{d}s\ \dot{\alpha}_n(s)$$

gegeben ist.
Hinweis: Damit Sie die späteren Teilaufgaben auf jeden Fall bearbeiten können, finden Sie alle notwendigen Zwischenergebnisse am Ende dieser Aufgabe.

Given the initial condition $\alpha(0) = 0$, the recursion equation is,

$$\alpha_{n+1}(t) = \int_0^t \dot{\alpha}_n(s)\ ds$$

$\alpha(0) = 0$ also implies,

$$v(t) = 0, \qquad q(0) = 0$$

Given $\alpha(0) = 0$, we calculate $\dot{\alpha}(0)(t)$ from the Langevin equation,

$$\dot{\alpha}_0(t) = \frac{\eta}{m}\cos(\omega t)\begin{pmatrix}0\\1\end{pmatrix}$$

Based on this result, we iterate for the first time to find $\alpha_1(t)$ using the recursion equation,

$$\alpha_1(t) = \int_0^t \frac{\eta}{m}\cos(\omega s)\begin{pmatrix}0\\1\end{pmatrix} ds = \frac{\eta}{m\omega}\sin(\omega t)\begin{pmatrix}0\\1\end{pmatrix}$$

[1] Diese ist auch als *Brownsche Bewegung* bekannt, die unter anderem von Einstein in einer seiner ersten Publikationen diskutiert wurde.

In order to perform the next iteration for finding $\alpha_2(t)$, we takes $\alpha_1(t)$ as the initial condition,

$$q_1(t) = 0, \qquad v_1(t) = \frac{\eta}{m\omega}\sin(\omega t)$$

, and $\dot{\alpha}_1(t)$ is again obtained from the Langevin equation,

$$\dot{\alpha}_1(t) = \begin{pmatrix} 0 & 1 \\ \eta/m & -\lambda/m \end{pmatrix} \begin{pmatrix} \cos(\omega t) \\ (\eta/m\omega)\sin(\omega t) \end{pmatrix} = \frac{\eta}{m}\cos(\omega t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\eta}{m\omega}\sin(\omega t)\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix}$$

Then the recursion equation gives,

$$\alpha_2(t) = \int_0^t \dot{\alpha}_1(s)\, ds = \alpha_1(t) + \frac{\eta}{m\omega^2}\Big[1 - \cos(\omega t)\Big]\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix}$$

Lastly, to find $\alpha_3(t)$, we first consider the initial condition given by $\alpha_2(t)$,

$$q_2(t) = \frac{\eta}{m\omega^2}\left[1 - \cos(\omega t)\right], \qquad v_2(t) = \frac{\eta}{m\omega}\sin(\omega t) - \frac{\lambda\eta}{(m\omega)^2}\left[1 - \cos(\omega t)\right]$$

and $\dot{\alpha}_2(t)$,

$$\dot{\alpha}_2(t) = \begin{pmatrix} 0 & 1 \\ \eta/m & -\lambda/m \end{pmatrix} \begin{pmatrix} \cos(\omega t - kq_2(t)) \\ (\eta/m\omega)\sin(\omega t) - (\lambda\eta/(m\omega)^2)\left[1 - \cos(\omega t)\right] \end{pmatrix}$$

Arranging the matrix equation,

$$\begin{aligned} \dot{\alpha}_2(t) = {} & \frac{\eta}{m}\cos(\omega t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\eta}{m\omega}\sin(\omega t)\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix} \\ & - \frac{\lambda\eta}{(m\omega)^2}[1 - \cos(\omega t)]\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix} + \frac{\eta}{m}\Big[\cos(\omega t - kq_2(t)) - \cos(\omega t)\Big]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Again, $\alpha_3(t)$ is obtained through the recursion equation. Integrating the first two terms in the above equation gives $\alpha_2(t)$,

$$\alpha_3(t) = \alpha_2(t) + \int_0^t -\frac{\lambda\eta}{(m\omega)^2}[1 - \cos(\omega s)]\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix} + \frac{\eta}{m}\Big[\cos(\omega s - kq_2(s)) - \cos(\omega s)\Big]\begin{pmatrix} 0 \\ 1 \end{pmatrix}\, ds$$

After integration,

$$\alpha_3(t) = \alpha_2(t) + \frac{\lambda\eta}{m^2\omega^3}\Big[\sin(\omega t) - \omega t\Big]\begin{pmatrix} 1 \\ -\lambda/m \end{pmatrix} + \frac{\eta}{m\omega}I(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where

$$I(t) = \frac{1}{\omega}\int_0^t \cos\left(\omega s - \frac{k\eta}{m\omega^2}\Big[1 - \cos(\omega t)\Big]\right) - \cos(\omega s)\, ds$$

(iv) Skizzieren Sie $\alpha_1(t)$ sowie $\alpha_2(t)$ im (Geschwindigkeits-)Phasenraum für verschiedene Werte von $\dfrac{\lambda}{m\omega}$.

Recall that, for $\alpha_1(t)$ and $\alpha_2(t)$,

$$q_1(t) = 0, \qquad v_1(t) = \frac{\eta}{m\omega}\sin(\omega t)$$

$$q_2(t) = \frac{\eta}{m\omega^2}\left[1 - \cos(\omega t)\right], \qquad v_2(t) = \frac{\eta}{m\omega}\sin(\omega t) - \frac{\lambda\eta}{(m\omega)^2}\left[1 - \cos(\omega t)\right]$$

The speed phase space sketched in Figure 2.

Um ein Gefühl für den Anwendungsbereich sowie die Genauigkeit der sukzessiven Approximation zu entwickeln, wollen wir nun die von Ihnen gefunden Lösungen genauer untersuchen. Aus der Vorlesung ist Ihnen bekannt, dass die Methode der sukzessiven Approximation *lokale* Lösungen liefert, d.h. Lösungen, die für hinreichend kleine t eine gute Approximation darstellen.

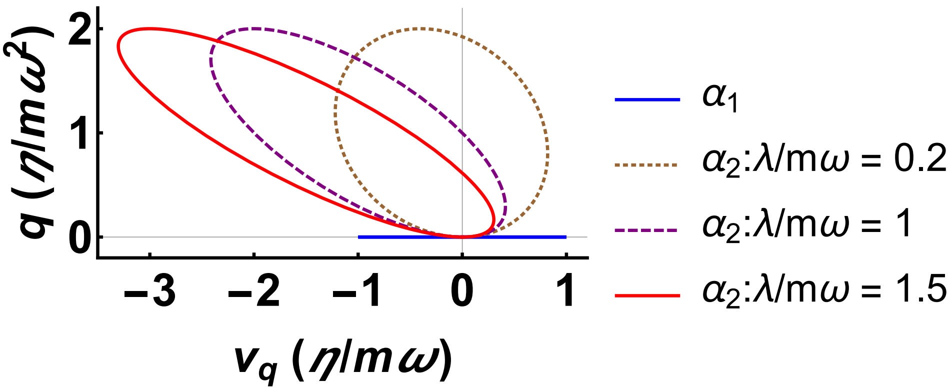


Abbildung 2: Parametric plot of phase space for $\alpha_1(t)$ and α_2(t)

(v) Argumentieren Sie ausgehend von den expliziten Ausdrücken für $\alpha_{1,2,3}$, dass die Bedingung eines *hinreichend kleinen* t in dieser Aufgabe gegeben ist durch $t \ll \frac{1}{\omega}$.
Hinweis: Hier ist noch keine Rechnung notwendig.

From $\alpha_{1,2,3}$, time dependence of $q(t)$ and $v(t)$ is always accompanied by ω in terms of $\cos(\omega t)$ or $\sin(\omega t)$. When the dynamical system evolves forward in time, starting from initial time $t_0 = 0$, the phase of the system is changed by ωt. Based on this observation, describing a dynamical system that deviates slightly from its initial configuration, is to consider a small change in phase,

$$\omega t \ll 1$$

This defines what we call 'small time'. The condition $t \ll 1/\omega$ defines the order of magnitude of time t, so that time evolution does not alternate significantly the phase of the system, so as its configuration.

(vi) Zeigen Sie durch explizite Rechnung, dass die Ihnen bekannten Glieder der Reihe $\{\alpha_n\}_n$ die folgenden Relationen erfüllen:

$$\alpha_1(t) = (\dots) \cdot (\omega t) + \mathcal{O}\big((\omega t)^2\big), \tag{3}$$

$$\alpha_2(t) - \alpha_1(t) = (\dots) \cdot (\omega t)^2 + \mathcal{O}\big((\omega t)^3\big), \tag{4}$$

$$\alpha_3(t) - \alpha_2(t) = (\dots) \cdot (\omega t)^3 + \mathcal{O}\big((\omega t)^4\big) \tag{5}$$

$(\dots)$ bezeichne hier jeweils unterschiedliche, von t unabhängige Koeffizienten. Was bedeuten diese Relationen für die Genauigkeit der sukzessiven Approximation für hinreichend kleine t?

Recall that $\alpha_{1,2,3}(t)$ has the following form:

$$\alpha_1(t) = \frac{\eta}{m\omega}\sin(\omega t)\begin{pmatrix}0\\1\end{pmatrix}$$

$$\alpha_2(t) = \alpha_1(t) + \frac{\eta}{m\omega^2}\Big[1 - \cos(\omega t)\Big]\begin{pmatrix}1\\-\lambda/m\end{pmatrix}$$

$$\alpha_3(t) = \alpha_2(t) + \frac{\lambda\eta}{m\omega^3}\Big[\sin(\omega t) - \omega t\Big]\begin{pmatrix}1\\-\lambda/m\end{pmatrix} + \frac{\eta}{m\omega}I(t)\begin{pmatrix}0\\1\end{pmatrix}$$

First of all, we expand $\sin(\omega t)$ and $\cos(\omega t)$ using Talyor series,

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}, \qquad \cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Then for $\alpha_1(t)$,

$$\alpha_1(t) = \frac{\eta}{m\omega}\Big[\,\omega t - \frac{(\omega t)^3}{3!} + \dots\Big]\begin{pmatrix}0\\1\end{pmatrix}$$

For $\alpha_2(t) - \alpha_1(t)$,

$$\alpha_2(t) - \alpha_1(t) = \frac{\eta}{m\omega^2}\Big[1 - \cos(\omega t)\Big]\begin{pmatrix}1 \\ -\lambda/m\end{pmatrix}$$
$$= \frac{\eta}{m\omega^2}\Big[\frac{(\omega t)^2}{2!} - \frac{(\omega t)^4}{4!} + ...\Big]\begin{pmatrix}1 \\ -\lambda/m\end{pmatrix}$$

For $\alpha_3(t) - \alpha_2(t)$,

$$\alpha_3(t) - \alpha_2(t) = \frac{\lambda\eta}{m\omega^3}\Big[\sin(\omega t) - \omega t\Big]\begin{pmatrix}1 \\ -\lambda/m\end{pmatrix} + \frac{\eta}{m\omega} I(t)\begin{pmatrix}0 \\ 1\end{pmatrix}$$
$$= \frac{\lambda\eta}{m\omega^3}\Big[-\frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{t!} + ...\Big]\begin{pmatrix}1 \\ -\lambda/m\end{pmatrix} + \frac{\eta}{m\omega} I(t)\begin{pmatrix}0 \\ 1\end{pmatrix}$$

where $I(t)$,

$$I(t) = \frac{1}{\omega}\int_0^t \cos\left(\omega s - \frac{k\eta}{m\omega^2}\Big[1 - \cos(\omega t)\Big]\right) - \cos(\omega s)\ ds$$

and expanding the integrand $\mathcal{I}(t)$ also with Taylor series,

$$\mathcal{I}(t) = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}f(t)^{2n} - \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(\omega t)^{2n}$$

where

$$f(t) = \omega t + \frac{k\eta}{m\omega^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(\omega t)^{2n}$$

Integrate over the lowest order of the integrand $\mathcal{I}(t)$,

$$I_0(t) = \int_0^t \mathcal{I}_0(s)\ ds = \int_0^t -\frac{1}{2}\Big[\omega s + \frac{k\eta}{m\omega^2}(\omega s)^2\Big]^2 + \frac{(\omega s)^2}{2}\ ds$$

Without further calculation, one can see that the lowest order of $I(t)$ is at least $\mathcal{O}(\omega t)^3$.

The most intutitive way to understand how they are related to the precision of the successive approximation is to consider them as a Taylor series. First of all, we showed the relation that,

$$(\alpha_{n+1} - \alpha_n) = c_n(\omega t)^n + \mathcal{O}(\omega t)^{n+1}$$

When we consider only small time, $\omega t \ll 1$, we can consider only the leading order with great accuracy, i.e.,

$$(\alpha_{n+1} - \alpha_n) \sim c_n(\omega t)^n$$

Also notice that,

$$\alpha_m(t) + \alpha_0 = \sum_{n=0}^{m-1}[\alpha_{n+1}(t) - \alpha_n(t)]$$

Therefore, when $\omega t \ll 1$, α_m can be written as,

$$\alpha_m(t) = \sum_{n=0}^{m-1}[\alpha_{n+1}(t) - \alpha_n(t)] = -\alpha_0 + \sum_{n=1}^{m-1}c_n(\omega t)^n\alpha_0$$

We can see that α_m is similar to a Taylor series which considers powers of ωt up to the power of m. By considering more high order terms, the approximation given by the Taylor series have a greater accuracy. Here we have a situation similar to a Taylor series, the only difference is that here we are constructing a series to resemble the exact solution of the given differential equation. The high order of m (iterations) we have, the more accurate of our iterative solution compared to the exact solution.

(vii) Entwickeln Sie $\alpha_1(t)$ sowie $\alpha_2(t)$ als Taylor-Reihe bis zur quadratischen Ordnung in ωt. Bestimmen Sie die Zeiten $t = t_v$ bzw. $t = t_q$, für die gilt, dass $|v_1(t_v)| = |v_2(t_v) - v_1(t_v)|$ bzw. $|q_1(t_q)| = |q_2(t_q) - q_1(t_q)|$. Welche Bedeutung haben t_q und t_v für die Genauigkeit der Approximation? Überprüfen Sie Ihre Interpretation anhand Ihrer Skizzen aus Teilaufgabe (iv).

Recall $\alpha_1(t)$ and $\alpha_2(t)$,

$$\alpha_1(t) = \frac{\eta}{m\omega}\left[\omega t - \frac{(\omega t)^3}{3!} + ...\right]\begin{pmatrix}0\\1\end{pmatrix}$$

$$\alpha_2(t) = \alpha_1(t) + \frac{\eta}{m\omega^2}\left[\frac{(\omega t)^2}{2!} - \frac{(\omega t)^4}{4!} + ...\right]\begin{pmatrix}1\\-\lambda/m\end{pmatrix}$$

At $t = t_v$ $|v_1(t_v)| = |v_2(t_v) - v_1(t_v)|$, t_v is given by the following relation,

$$\frac{\eta}{m\omega}\left[\omega t_v - \frac{(\omega t_v)^3}{3!} + ...\right] = \frac{\eta\lambda}{(m\omega)^2}\left[\frac{(\omega t_v)^2}{2!} - \frac{(\omega t_v)^4}{4!} + ...\right]$$

Consider only up to quadratic order,

$$\frac{\eta}{m\omega}(\omega t_v) = \frac{\eta\lambda}{2m^2\omega^2}(\omega t_v)^2$$

Then, for $|v_1(t_v)| = |v_2(t_v) - v_1(t_v)|$, we have,

$$t_v = 0, \qquad t_v = \left|\frac{2m}{\lambda}\right|$$

On the other hand, at $t = t_q$, $|q_1(t_q)| = |q_2(t_q) - q_1(t_q)|$,

$$0 = \frac{\eta}{m\omega^2}\frac{(\omega t)^2}{2!}$$

This impies,

$$t_q = 0 \tag{6}$$

The times t_v and t_q are the times until which $\alpha_1(t)$ is a good approximation for the velocity/position, in the sense that the changes obtained by moving to α_2 are small in comparison. Let us first observe that the difference between α_2 and α_1 is $\alpha_2(t) - \alpha_1(t)$. So, as long as this term is significantly smaller than $\alpha_1(t)$, α_2 is related to α_1 by only a small correction. A good measure when this breaks down is to set them equal: $\alpha_2(t) - \alpha_1(t)$ being as large as $\alpha_1(t)$ means that the corrections leading to $\alpha_2(t)$ are as large as $\alpha_1(t)$, so that they can no longer be ignored.
Since α consists of two components - the velocity and the position - each of these components yields a different time. We find that the assumption that α_1 yields a good approximation for the velocity breaks down at t_v. Interestingly, we also find that the α_1-contribution to the position breaks down immediately, and the leading order contribution is given by the term arising from α_2.

Zwischenergebnisse zu Aufgabe 2

$$\alpha_1(t) = \frac{\eta}{m\omega}\cdot\begin{pmatrix}0\\ \sin\omega t\end{pmatrix}, \tag{7}$$

$$\alpha_2(t) = \alpha_1(t) + \big(1-\cos(\omega t)\big)\cdot\frac{\eta}{m\omega}\cdot\begin{pmatrix}1\\ -\frac{\lambda}{m\omega}\end{pmatrix}, \tag{8}$$

$$\alpha_3(t) = \alpha_2(t) + \frac{\eta}{m\omega}\cdot\begin{pmatrix}\frac{\lambda}{m\omega^2}\big(\sin(\omega t) - \omega t\big)\\ -\frac{\lambda^2}{m^2\omega^2}\cdot\big(\sin(\omega t) - \omega t\big) + I(t)\end{pmatrix}, \tag{9}$$

wobei

$$I(t) = \omega\int_0^t \mathrm{d}s\ \cos\left(\omega s - \frac{k\eta}{m\omega^2}(1-\cos\omega t)\right) - \cos(\omega s) \tag{10}$$

3 Kepler Problem - Teil 2

Wir betrachten erneut ein Teilchen der Masse m, welches sich in einem Zentralpotential bewege.

(i) Eine andere Herangehensweise an Zentralpotentiale ist die Betrachtung des Azimuthwinkels φ und wie dieser von r abhängt. Zeigen Sie hierfür

$$\varphi = \varphi_0 + \int \frac{(L/mr^2)}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}} \mathrm{d}r \tag{11}$$

wobei L den Drehimpuls bezeichnet und U_{eff} das effektive Potential.
Hinweis: Nutzen Sie die Kettenregel um aus der Bewegungsgleichung $\frac{\mathrm{d}\varphi}{\mathrm{d}r}$ *zu bestimmen.*

Recall from the last exercise that,

$$E = \frac{1}{2}m\dot{r}^2(t) + U_{\text{eff}}(r)$$

and we also solved for $\dot{r}$,

$$\dot{r}(t) = \sqrt{\frac{2}{m}\Big[E - U_{\text{eff}}(r)\Big]}$$

Using chain rule, we have,

$$\dot{\varphi}(t)\frac{dr}{d\varphi} = \sqrt{\frac{2}{m}\Big[E - U_{\text{eff}}(r)\Big]}$$

the angular velocity can be expressed in terms of the angular momentum L, which is an invariant in our current scenario,

$$L = mr^2\dot{\varphi}$$

then we have

$$\frac{d\varphi}{dr} = \frac{L/(mr^2)}{\sqrt{(2/m)[E - U_{\text{eff}}(r)]}}$$

Integrate the entire equation by r, we recover equation (10),

$$\int_{\varphi_0}^{\varphi} d\varphi = \int_{r_0}^{r(t)} \frac{L/(mr^2)}{\sqrt{(2/m)[E - U_{\text{eff}}(r)]}}\, dr$$

(ii) Wir betrachten den Fall einer ungebundenen Lösung eines Teilchens im Potential $U(r) = -\frac{\alpha}{r}$. Berechnen Sie die Zeitabhängigkeit der Teilchenkoordinaten für $E = 0$. Skizzieren und diskutieren Sie Ihre Lösung.

Recall that the effective potential,

$$U_{\text{eff}} = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} - \frac{\alpha}{r}$$

Now we consider the integration from the previous chapter, with $E = 0$ and $U(r) = -\alpha/r$,

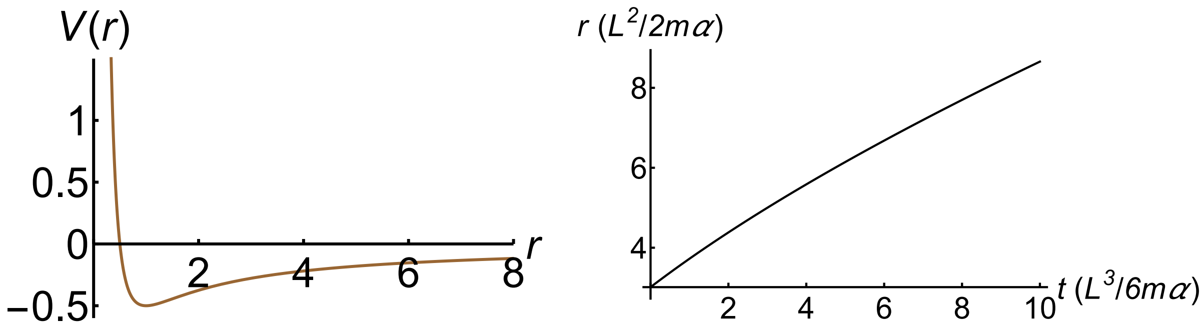
$$t - t_0 = \int_{r_0}^{r(t)} \left[\frac{2}{m}\left(\frac{\alpha}{r} - \frac{L^2}{2mr^2}\right)\right]^{-1/2} dr = \sqrt{\frac{m}{2\alpha}} \int_{r_0}^{r(t)} \frac{rdr}{\sqrt{r - (L^2/2m\alpha)}}$$

By the following change of variable, and let,

$$a \equiv \frac{L^2}{2m\alpha}, \qquad x = r - a, \qquad \frac{dx}{dr} = 1$$

It becomes,

$$t - t_0 = \sqrt{\frac{m}{2\alpha}} \int_{x_0}^{x(t)} \frac{x + a}{\sqrt{x}}\, dx = \left(\sqrt{\frac{mx(t)}{2\alpha}}\left[\frac{2x(t)}{3} + 2a\right]\right)_{x_0}^{x(t)}$$



(a) The effective potential $U_{\text{eff}}(r)$ (b) Radial motion of the particle $r(t)$

Abbildung 3: Sketch of motion and potential energy profile

Collecting together all the integration constants,

$$t = \sqrt{\frac{2mx(t)}{\alpha}}\left[\frac{x(t)}{3} + a\right] + t_0$$

Since $E = 0$, the particle's kinetic energy equals to its potential energy. Therefore it has sufficient energy to escape towards $r = \infty$. Another possible motion which is not covered in Figure 3(b) is that, the particle can move radially inwards, bounced back by the $1/r$ potential and fly off to infinity.

(iii) Betrachten Sie erneut Gleichung (11). Es sei $r_{\text{max}} \geq r \geq r_{\text{min}}$. Bestimmen Sie hieraus eine Gleichung für die Änderung des Winkels während eines Umlaufs von r_{min} nach r_{max} und retour. Welche Bedingung muss erfüllt sein, damit die Bewegung einer geschlossenen Bahn folgt? Um welchen Winkel $\delta\varphi$ verschiebt sich das Perihel der Bewegung pro Umlauf, wenn das Potential eine kleine Störung $\frac{\beta}{r^2}$ erfährt? *Hinweis: Entwickeln Sie das Integral für $U = -\frac{\alpha}{r} + \delta U$ in erster Ordnung in δU und setzen Sie anschließend $\delta U = \frac{\beta}{r^2}$.*

We first look into $\Delta\varphi(U_0)$,

$$\varphi - \varphi_0 = \int_{r(\varphi_0)}^{r(\varphi)} \sqrt{\frac{m}{2}}\frac{L}{mr^2}\frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} = \int_{r(\varphi_0)}^{r(\varphi)} \frac{L}{r^2}\frac{dr}{\sqrt{2mE + (2m\alpha/r) - (L^2/r^2)}}$$

Consider a change of variable,

$$u = \frac{L}{r}, \qquad \frac{du}{dr} = -\frac{L}{r^2}$$

Then we have,

$$\varphi - \varphi_0 = -\int_{u(\varphi_0)}^{u(\varphi)} \frac{du}{\sqrt{2mE + (2m\alpha/L)u - u^2}} = \arctan\left(\frac{(m\alpha/L) - u}{\sqrt{2mE + (2m\alpha/L)u - u^2}}\right)_{u(\varphi_0)}^{u(\varphi)}$$

In brief, the above integration is done by:

$$\varphi - \varphi_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{a + bu - u^2}} = -\int_{u_0}^{u} \frac{du}{\sqrt{a + (b^2/4) - [(b/2) - u]^2}}$$

And a change of variable $v = b/2 - u$, and $c^2 = a + (b^2/4)$,

$$\varphi - \varphi_0 = \int_{v_0}^{v} \frac{dv}{\sqrt{c^2 - v^2}} = \arctan\left(\frac{v}{\sqrt{c^2 - v^2}}\right)_{v_0}^{v}$$

Before we proceed to solve for $u(\varphi)$ explicitly, now we explore the physical condition imposed by the boundary at $r = r_{\text{max}}$ and $r = r_{\text{min}}$, where $r_{\text{max/min}}$ are the extreme radius of the orbit. Since $r_{\text{min/max}}$ is the extreme radius of the motion, we expect $\dot{r} = 0$ at $r = r_{\text{min/max}}$. This implies that the energy at $r = r_{\text{min/max}}$ contains only potential energy:

$$E = U_{\text{eff}} = -\frac{\alpha}{r} + \frac{L^2}{2mr^2}$$

such that,

$$2mE = -\frac{2m\alpha}{L}u + u^2$$

Then we can solve for $u_{\text{max/min}}$,

$$u = \frac{m\alpha}{L} \pm \frac{m\alpha}{L}\sqrt{1 + \frac{2EL^2}{m\alpha^2}}$$

and requiring $r_{\text{max}} > r_{\text{min}}$ implies $u_{\text{max}} < u_{\text{min}}$ becuase $u = l/r$. In detail,

$$u_{\text{max}} = \frac{m\alpha}{L}\left(1 - \sqrt{1 + \frac{2EL^2}{m\alpha^2}}\right), \qquad u_{\text{min}} = \frac{m\alpha}{L}\left(1 + \sqrt{1 + \frac{2EL^2}{m\alpha^2}}\right),$$

Substitute $u_{\text{max/min}}$ into our integration equation, it fixes the boundary term as follow,

$$\varphi - \varphi_0 = \arctan\left(\frac{(m\alpha/L)\sqrt{1 + (2EL^2/m\alpha^2)}}{\sqrt{2mE + (2m\alpha/L)u - u^2}}\right) - \arctan\left(-\frac{(m\alpha/L)\sqrt{1 + (2EL^2/m\alpha^2)}}{\sqrt{2mE + (2m\alpha/L)u - u^2}}\right)$$

Also for,

$$2mE = -(2m\alpha/L)u + u^2, \quad \Longrightarrow \quad 2mE + (2m\alpha/L)u - u^2 = 0$$

so that the denominator inside the argument of arctan becomes zero, so we have

$$\varphi - \varphi_0 = \lim_{x\to\infty}\Big[\arctan(x) - \arctan(-x)\Big] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

This tells us that, when the radius vector varies from r_{min} to r_{max}, the azimuth angle is changed by π. Now we want to ask if the orbit is a closed path. The condition of a closed path is that: when $r(t)$ varies from r_{min} to r_{max}, and continue towards r_{min}, the change of azimuth angle is 2π, this is: $\Delta\varphi = 2\pi$. This means that the radius vector will return to its initial position after making a complete revolution. This can be see from varying r from r_{min} to r_{max} and back, which gains only an extra factor of 2,

$$\varphi - \varphi_0 = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{\frac{m}{2}}\frac{L}{mr^2}\frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} = 2\pi$$

Now we want to disturb the potential $U(r)$ by $U(r) = U_0(r) + \delta U(r)$, where $\delta U(r) = \beta/r^2$. When the potential experiences a small disturbance, the angle $\Delta\varphi$ is shifted from 2π and the orbital does not have in general a closed path,

$$\Delta\varphi[U_0 + \delta U] \approx \Delta\varphi[U_0] + \delta\varphi[U]$$

We are interested in calculating the perturbation $\delta\varphi[U]$. To begin with, firstly we introduce the Leibniz integration rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)dt$$

We want to rewrite it in terms of our integral:

$$\varphi - \varphi_0 = \int_{r(\varphi_0)}^{r(\varphi)} \sqrt{\frac{m}{2}}\frac{L}{mr^2}\frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}$$

Firstly we treat the angular momentum L as variable l, notice that,

$$\int_{r(\varphi_0)}^{r(\varphi)} \frac{d}{dl}\sqrt{E - U_{\text{eff}}(r)}\ dr = \int_{r(\varphi_0)}^{r(\varphi)} -\frac{l}{2mr^2}\frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} = -\frac{1}{\sqrt{2m}}(\varphi - \varphi_0)$$

From the Leibniz integration rule, it implies,

$$\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{d}{dl}\sqrt{E - U_{\text{eff}}(r)}\ dr = \frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{E - U_{\text{eff}}(r)}\ dr$$
$$+ \sqrt{E - U_{\text{eff}}(r_{\text{min}})}\frac{d}{dl}r_{\text{min}}(l) - \sqrt{E - U_{\text{eff}}(r_{\text{max}})}\frac{d}{dl}r_{\text{max}}(l)$$

Again we use the condition that: $\dot{r}_{\text{max/min}} = 0$ at $r = r_{\text{max/min}}$. The kinetic energy is zero implies,

$$\sqrt{E - U_{\text{eff}}(r_{\text{max/min}})} = 0$$

The Leibniz integration reduces to the following relation,

$$\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{d}{dl}\sqrt{E - U_{\text{eff}}(r)}\, dr = \frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{E - U_{\text{eff}}(r)}\, dr$$

where the left side of the above equation is the integration that we are interested at the beginning. Now, we begin to compute $\delta\varphi$, first of all, we have,

$$\Delta\varphi[U] = -\frac{1}{\sqrt{2m}}\frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{E - U_{\text{eff}}(r)}\, dr$$

For our convenient, we define,

$$U(r) = U_0(r) + \delta U(r), \qquad U_{\text{eff}} = U_0(r) + \frac{l^2}{2mr^2}$$

where $U_0(r) = -\alpha/r$. Since $\delta U(r)$ is a tiny disturbance potential,

$$\delta U(r) \ll U_0(r)$$

This allows us to expand the integrand as,

$$\sqrt{E - U_{\text{eff}}(r) - \delta U(r)} = \sqrt{E - U_{\text{eff}}(r)} + \frac{\delta}{\delta U(r)}\sqrt{E - U_{\text{eff}}(r) - \delta U(r)}\Big|_{\delta U(r)=0}\delta U(r)$$

So that the change in azimuth angle can be separated into two terms,

$$\begin{aligned}\Delta\varphi[U] &= -2\sqrt{2m}\frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{E - U_{\text{eff}}(r)}\, dr + 2\sqrt{2m}\frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{d}{dl}\left[\frac{\delta U(r)}{\sqrt{E - U_{\text{eff}}(r)}}\right] dr \\ &= 2\pi + \sqrt{2m}\frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \left[\frac{\delta U(r)}{\sqrt{E - U_{\text{eff}}(r)}}\right] dr\end{aligned}$$

The first term is the unperturbed motion; while the second term describes how the azimuth angle is shifted, $\delta\varphi[U]$, by perturbation $\delta U(r)$,

$$\delta\varphi[U] = \sqrt{2m}\frac{d}{dl}\int_{r_{\text{min}}}^{r_{\text{max}}} \left[\frac{\delta U(r)}{\sqrt{E - U_{\text{eff}}(r)}}\right] dr$$

Notice that our previous result provides the relation:

$$\frac{d\varphi}{dr} = \frac{l/(mr^2)}{\sqrt{(2/m)[E - U_{\text{eff}}(r)]}}$$

Then $\delta\varphi$ is simplified as,

$$\delta\varphi[U] = \frac{d}{dl}\int_0^{\pi} \frac{2mr^2}{l}\delta U(r)\, d\varphi = \frac{d}{dl}\int_0^{\pi} \frac{2m\beta}{l}\, d\varphi$$

The perturbation is therefore,

$$\delta\psi[U] = -\frac{2\pi m\beta}{l^2}$$