Prof. Dr. Stefan Hofmann Sommersemester 2020

Übungen zur Theoretischen Mechanik (T1)

Blatt 3 - Lösung

Dieses Blatt befasst sich mit kovarianten Ableitungen, Christoffelsymbolen und deren Bedeutung für die Beschreibung von Scheinkräften. Im Sinne der Übersichtlichkeit werden wir für die Koordinatendarstellung von Kurven die Notation $q^a(t) = x^a(\gamma(t))$ verwenden. Sämtliche Indizes sind Elemente der Indexmenge $I = \{1, 2, 3\}$, außerdem wird konsequent die Einsteinsche Summenkonvention verwendet.

1 Allgemeine Scheinkräfte

(i) Das zweite Newtonsche Gesetz lautet $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$, wobei $\boldsymbol{x}$ den Ortsvektor im euklidischen Vektorraum als Teil der Galilei-Raumzeit bezeichnet. Zeigen Sie, dass dieses Gesetz in einer beliebigen, **nicht** explizit zeitabhängigen Basis die folgende Form annimmt:

$$\frac{1}{m}F^a\big(q(t)\big) = \ddot{q}^a(t) + \Gamma^a_{bc}\big(q(t)\big)\dot{q}^b(t)\dot{q}^c(t) \tag{1}$$

Wiederholen Sie dabei insbesondere die Herleitung des Ausdrucks für die Beschleunigung, der Sie in der Vorlesung begegnet sind.

First of all, from concept 1.2.5, and 1.2.6, we know that the velocity vector $\dot{\gamma}(t)$ can be expressed in terms of $x^a(\gamma(t))$ by taking the directional derivative along the curve $\gamma(t)$,

$$\dot{\gamma}(t) = \sum_{a=1}^{3} \dot{x}^a(\gamma(t))e_a(\gamma(t)) = \sum_{a=1}^{3} \dot{q}^a(t)e_a(\gamma(t))$$

For convenience, we adopt a simpler notation, $e_a(\gamma(t)) \equiv e_a$, and $\sum_{a=1}^{3} \equiv \sum_a$.
In order to find the the coordinate representation of the acceleration, we take the directional derivative of the vector field $\dot{\gamma}(t)$ (equation 1.17),

$$\ddot{\gamma}(t) = \sum_a \Big(\ddot{q}^a(t)e_a + \dot{q}^a(t)\frac{d}{dt}e_a\Big) = \sum_a \Big[\ddot{q}^a(t)e_a + \dot{q}^a(t)\Big(\sum_b \dot{q}^b(t)\partial_b e_a\Big)\Big]$$

As we have seen in the lecture notes, the Christoffel Symbols are defined in Concept 1.2.8 by

$$D_{e_a}e_b = \Gamma^c_{ab}e_c$$

and therefore we have

$$\partial_a e_b = \Gamma^c_{ab}e_c$$

Inserting the Christoffel symbol into the coordinate representation of the acceleration then yields

$$\ddot{\gamma}(t) = \sum_a \Big[\ddot{q}^a(t)e_a + \sum_b\sum_c \dot{q}^a(t)\dot{q}^b(t)\Gamma^c_{ab}e_c\Big]$$

Then, we relabel the indices,

$$\ddot{\gamma}(t) = \sum_a \Big[\ddot{q}^a(t)e_a + \sum_b\sum_c \dot{q}^b(t)\dot{q}^c(t)\Gamma^a_{bc}e_a\Big]$$

It becomes obvious that, if we consider only the components, at the same time adopting the Einstein summation convention, we can reconstruct equation (1).

$$\frac{1}{m}F^a(q(t)) = \ddot{q}^a(t) + \Gamma^a_{bc}\dot{q}^b(t)\dot{q}^c(t),$$

where we have been using a convention $\Gamma^a_{bc} \equiv \Gamma^a_{bc}(\gamma(t))$.

(ii) Der zweite der Terme auf der rechten Seite von (1) kann als eine *Scheinkraft* interpretiert werden. Wie unterscheidet sich diese Scheinkraft von einer realen Kraft?

The second term on the right side of equation (1) represents the fictitious force. It appears when one describes inertial motion in non-inertial reference frame.
By comparison, the *real* force arises from interaction between two objects, such as electromagnetic force; while the fictitious force comes from the acceleration of the non-inertial reference frame.
In other words, from the perspective of the non-intertial frame, an object under inertial motion appears to be accelerated by a force. That being said, the fictitious forces can be removed by changing the coordinate system as well.

(iii) Betrachten Sie nun eine explizit zeitabhängige Basis, d.h. Basisvektoren der Form $\boldsymbol{e}_a = \boldsymbol{e}_a\big(q(t), t\big)$. Zeigen Sie, dass sich das Newtonsche Gesetz in einer solchen Basis wie folgt schreiben lässt:

$$\frac{1}{m}F^a\big(q(t)\big) = \ddot{q}^a(t) + \Gamma^a_{bc}\big(q(t)\big)\dot{q}^b(t)\dot{q}^c(t) + 2\Lambda^a_b\big(q(t)\big)\big)\dot{q}^b(t) \tag{2}$$

Geben Sie die Koeffizienten Λ^a_b explizit an.
Hinweis: Überlegen Sie sich zunächst, dass die Zeitableitung eines Basisvektors dieser Basis entlang der Kurve γ gegeben ist durch

$$\frac{d}{dt}\boldsymbol{e}_a\big(q(t), t\big) = \dot{q}^b(t)\partial_b\boldsymbol{e}_a\big(q(t), t\big) + \frac{\partial \boldsymbol{e}_a}{\partial t}\big(q(t), t\big). \tag{3}$$

In this question, we have only modified the basis vector by an additional, explicit time dependence. Therefore we start from the intermediate step of question (i),

$$\ddot{\gamma}(t) = \ddot{q}^a(t)e_a + \dot{q}^a(t)\frac{d}{dt}e_a$$

where we have already implemented Einstein's summation convention. Now, from equation (3), we would have an extra term compared to question 1(i),

$$\ddot{\gamma}(t) = \ddot{q}^a(t)e_a + \Gamma^a_{bc}\,\dot{q}^b(t)\dot{q}^c(t)e_a + \dot{q}^a(t)\frac{\partial e_a}{\partial t}$$

In analogy to the Christoffel symbol,

$$\partial_a e_b = \Gamma^c_{ab}e_c$$

the coefficients Λ^b_a can be defined in terms of a time derivative of the basis vectors,

$$\partial_t e_a = 2\Lambda^b_a e_b \tag{4}$$

Then we would have,

$$\ddot{\gamma}(t) = \ddot{q}^a(t)e_a + \Gamma^a_{bc}\,\dot{q}^b(t)\dot{q}^c(t)e_a + 2\dot{q}^a(t){\Lambda_a}^b e_b$$

After relabelling our indices, and considering again only the components, we have,

$$\frac{1}{m}F^a(\gamma(t)) = \ddot{q}^a(t) + \Gamma^a_{bc}\,\dot{q}^b(t)\dot{q}^c(t) + 2\Lambda^a_b\dot{q}^b(t)$$

This is exactly equation (2). Taking the scalar product of (4) with an arbitrary basis vector, we find

$$\Lambda^b_a\langle e_b, e_c\rangle = \Lambda^b_a g_{bc} = \frac{1}{2}\big(\partial_t\langle e_a, e_c\rangle\big) = \frac{1}{2}\partial_t g_{ac}$$

Therefore Λ^b_a is

$$\Lambda^b_a = \frac{1}{2}g^{bc}\partial_t g_{ac}$$

(iv) Betrachten Sie nun eine Punktmasse, die einer konstanten Kraft ausgesetzt ist. Ist es möglich, ein Inertialsystem zu finden, in dem das Teilchen zu zwei **verschiedenen** Zeitpunkten $t_1 \neq t_2$ ruht? Begründen Sie ihre Antwort.

Firstly, a point mass subject to a constant force is described by

$$\frac{1}{m} F^a(\gamma(t)) = \ddot{q}^a(t)$$

This implies that this point mass experiences constant acceleration. In short, a point mass under constant force cannot be at rest after a finite time, for any inertia frame.
Firstly, an inertia frame is a frame of reference that is at rest or moving at constant velocity. If external force is non-zero, the point mass is accelerated, as described by the equation above. At any instant of time, we can find an inertial reference frame, which has the same velocity as the point mass and therefore seeing it at rest. However, due to the constant amount of force, the point mass at the next instant would accelerate and changes its velocity. Thus, after a finite amount of time, such inertial reference frame again will not see the point mass at rest anymore. This idea repeats whenever we attempt to find an inertia frame to see the point mass at rest. Therefore, when a point mass is under constant force, no inertia frame can describe it at rest after finite time.

(v) Gleichung (2) enthält zwei Terme, die Scheinkräfte beschreiben, und die sich beide auf die Wahl einer Basis zurückführen lassen. Ist es möglich, beide Scheinkräfte durch einen einzigen Ausdruck zu beschreiben? Wie muss die Theorie angepasst werden, um dies zu ermöglichen, und welche physikalische Interpretation könnte diese Modifikation haben?

To combine both fictitious force terms, we only need to extend our description of metric and Christoffel symbols to both space and time,

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \Big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \Big)$$

Let's consider for instance the case of one of the lower indices corresponding to the time t,

$$\Gamma^b_{ta} = \frac{1}{2} g^{bc} \Big(\partial_t g_{ac} + \partial_a g_{tc} - \partial_c g_{ta} \Big)$$

In Galileo spacetime, time as a one-dimensional oriented space evolves identically in all space, thus their basis vector should be orthogonal (or independent) to all the others,

$$g_{ti} = 0, \quad \text{for} \quad i = 1, 2, 3$$

Then we have,

$$\Gamma^b_{ta} = \frac{1}{2} g^{bc} \partial_t g_{ac} = \Lambda^b_a$$

which implies that the two fictitious terms are actually a single term when we extend our definition of the Christoffel symbols to cover the time dimension,

$$\begin{aligned} \frac{1}{m} F^a(\gamma(t)) &= \ddot{q}^a(t) + \Gamma^a_{bc}\, \dot{q}^b(t) \dot{q}^c(t) + 2\Lambda^a_b \dot{q}^b(t) \\ &= \ddot{q}^a(t) + \Gamma^a_{bc}\, \dot{q}^b(t) \dot{q}^c(t) + 2\Gamma^a_{bt}\, \dot{q}^b(t) \dot{q}^{(t)}(t) \end{aligned}$$

where $q^{(t)}(t) = dt/dt = 1$ in Galileo spacetime. Since the Christoffel symbol is symmetric on its lower indices,

$$2\Gamma^a_{bt}\, \dot{q}^b(t) \dot{q}^{(t)}(t) = \Gamma^a_{bt}\, \dot{q}^b(t) \dot{q}^{(t)}(t) + \Gamma^a_{tb}\, \dot{q}^{(t)}(t) \dot{q}^b(t)$$

so that, at last, we can absorb the second term of the fictitious force with the first term,

$$\frac{1}{m} F^a(\gamma(t)) = \ddot{q}^a(t) + \Gamma^a_{\mu\nu}\, \dot{q}^\mu(t) \dot{q}^\nu(t)$$

where μ and ν and are $1, 2, 3$ and t.

2 Zentrifugalkraft

(i) Bestimmen Sie zunächst die Metrik $g_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$ in der lokalen Basis, die sich aus der Wahl von Kugelkoordinaten ergibt. Nutzen Sie diese, um die inverse Metrik g^{ij}, die definiert ist durch

$$g^{im} g_{mj} = \delta^i_j, \tag{5}$$

zu finden.

Firstly we represent $\boldsymbol{x}$ in Cartesian coordinates, which will later be labelled by x^i, while y^i refers to the spherical cooridinates,

$$\frac{\partial \boldsymbol{s}}{\partial y^i} = \sum_j \frac{\partial x^j}{\partial y^i} \hat{e}_j$$

for example, say $y^i = r$,

$$\frac{\partial \boldsymbol{x}}{\partial r} = \frac{\partial x^1}{\partial r} \hat{e}_1 + \frac{\partial x^2}{\partial r} \hat{e}_2 + \frac{\partial x^3}{\partial r} \hat{e}_3$$

We know how to express the position vector in terms of spherical coordinates,

$$\boldsymbol{x} = \begin{bmatrix} r \sin\theta \cos\psi \\ r \sin\theta \sin\psi \\ r \cos\theta \end{bmatrix}$$

Then we can evaluate,

$$\frac{\partial \boldsymbol{x}}{\partial r} = \begin{bmatrix} \sin\theta \cos\psi \\ \sin\theta \sin\psi \\ \cos\theta \end{bmatrix}, \qquad \frac{\partial \boldsymbol{x}}{\partial \theta} = \begin{bmatrix} r \cos\theta \cos\psi \\ r \cos\theta \sin\psi \\ -r \sin\theta \end{bmatrix}, \qquad \frac{\partial \boldsymbol{x}}{\partial \psi} = \begin{bmatrix} -r \sin\theta \sin\psi \\ r \sin\theta \cos\psi \\ 0 \end{bmatrix},$$

Therefore the local basis is given by,

$$e_r = \frac{\partial \boldsymbol{x}}{\partial r}, \qquad e_\theta = \frac{\partial \boldsymbol{x}}{\partial \theta}, \qquad e_\psi = \frac{\partial \boldsymbol{x}}{\partial \psi}$$

The scalar products of these local basis vectors are,

$$\langle e_r, e_r \rangle = 1, \qquad \langle e_\theta, e_\theta \rangle = r^2, \qquad \langle e_\psi, e_\psi \rangle = r^2 \sin^2\theta,$$

$$\langle e_i, e_j \rangle = 0 \quad \text{for} \quad i \neq j$$

This means that g_{ij} in matrix representation is,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

and therefore its inverse is simply,

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2\theta) \end{pmatrix}$$

(ii) In der Vorlesung haben Sie die folgende Formel für die Christoffelsymbole kennengelernt:

$$\Gamma^c_{ab}(\boldsymbol{x}) = \frac{1}{2} g^{cj}(\boldsymbol{x}) \big(\partial_a g_{jb}(\boldsymbol{x}) + \partial_b g_{ja}(\boldsymbol{x}) - \partial_j g_{ab}(\boldsymbol{x}) \big) \tag{6}$$

Nutzen Sie diese, um die Christoffelsymbole des euklidischen Raums in Kugelkoordinaten zu bestimmen.

Hinweis: Überlegen Sie sich zunächst, welche Eigenschaften der Christoffelsymbole Sie nutzen können, *um die Rechnung zu verkürzen.*

Firstly, observe that the Christoffel symbols are symmetric under exchange of its lower indices,

$$\Gamma^a_{bc} = \Gamma^a_{cb}$$

We are very glad to take this into consideration, as it reduces the number Christoffel symbols that need to be calculated. Then, the non-zero Christoffel symbols in the spherical coordinate are the following,

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\psi\psi} = -r\sin^2\theta, \qquad \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta_{\psi\psi} = -\sin\theta\cos\theta,$$

$$\Gamma^\psi_{\psi r} = \frac{1}{r}, \qquad \Gamma^\psi_{\psi\theta} = \frac{1}{\tan\theta}$$

Alternatively, if one favours to write in compact notation, and showing all the zeros and symmetric components for no good reason, we can simply express the Christoffel symbol in matrix representation,

$$\Gamma^r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix}, \qquad \Gamma^\theta = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix},$$

$$\Gamma^\phi = \begin{pmatrix} 0 & 0 & 1/r \\ 0 & 0 & 1/\tan\theta \\ 1/r & 1/\tan\theta & 0 \end{pmatrix},$$

(iii) Vergewissern Sie sich, dass für die so bestimmten Christoffelsymbole tatsächlich

$$\partial_a \boldsymbol{e}_b(\boldsymbol{x}) = \Gamma^c_{ab}(\boldsymbol{x})\boldsymbol{e}_c(\boldsymbol{x}) \tag{7}$$

gilt.

Let's for example take $\Gamma^r_{\theta\theta}$, where this relation becomes

$$\partial_\theta e_\theta = \Gamma^r_{\theta\theta} e_r$$

On the left hand side, we have,

$$\partial_\theta e_\theta = \begin{bmatrix} -r\sin\theta\cos\psi \\ -r\sin\theta\sin\psi \\ -r\cos\theta \end{bmatrix}$$

On the right hand side,

$$\Gamma^r_{\theta\theta} e_r = -r \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix}$$

On a less trivial example,

$$\partial_\psi e_\psi = \Gamma^r_{\psi\psi} e_r + \Gamma^\theta_{\psi\psi} e_\theta$$

On the left side we have,

$$\partial_\psi e_\psi = \begin{bmatrix} -r\sin\theta\cos\psi \\ -r\sin\theta\sin\psi \\ 0 \end{bmatrix}$$

And the right side, there are two non-zero Christoffel symbols that we must include and sum over,

$$\Gamma^r_{\psi\psi} e_r + \Gamma^\theta_{\psi\psi} e_\theta = -r\sin^2\theta \begin{bmatrix} \sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta \end{bmatrix} - \sin\theta\cos\theta \begin{bmatrix} r\cos\theta\cos\psi \\ r\cos\theta\sin\psi \\ -r\sin\theta \end{bmatrix} = \begin{bmatrix} -r\sin\theta\cos\psi \\ -r\sin\theta\sin\psi \\ 0 \end{bmatrix}$$

The rest are just identical repetitive computation.

(iv) Betrachten Sie nun die Bahn einer Punktmasse in Kugelkoordinaten, $q(t) = (r(t), \theta(t), \phi(t))$. Bestimmen Sie die Scheinkräfte, die in diesem Koordinatensystem auf die Punktmasse wirken.

Recall that,

$$\frac{1}{m} F^a(\gamma(t)) = \ddot{q}^a(t) + \Gamma^a_{bc}\, \dot{q}^b(t)\dot{q}^c(t)$$

Let's for this moment forget about any *real* force, we put $\ddot{q}^a(t) = 0$. Along the direction e_r,

$$\begin{aligned}\frac{F^r(\gamma(t))}{m} &= -\Gamma^r_{\theta\theta}\dot{\theta}(t)^2 - \Gamma^r_{\psi\psi}\dot{\psi}(t)^2 \\ &= -r\,\dot{\theta}(t)^2 - r\sin^2\theta\,\dot{\psi}(t)^2\end{aligned}$$

For direction e_θ,

$$\frac{F^\theta(\gamma(t))}{m} = \frac{2}{r}\dot{\theta}(t)\dot{r}(t) - \sin\theta\cos\theta\dot{\psi}(t)^2$$

For direction e_ψ,

$$\frac{F^\psi(\gamma(t))}{m} = \frac{2}{r}\dot{\psi}(t)\dot{r}(t) + \frac{2}{\tan\theta}\dot{\psi}(t)\dot{\theta}(t)$$

3 Corioliskraft

In dieser Aufgabe untersuchen wir ein mit konstanter Winkelgeschwindigkeit ω rotierendes Koordinatensystem. Die entsprechenden Koordinaten (r, θ, ψ) seien wie folgt durch die kartesischen Koordinaten definiert:

$$x = r\cos(\psi - \omega t)\sin(\theta),\ \ y = r\sin(\psi - \omega t)\sin(\theta),\ \ z = r\cos(\theta). \tag{8}$$

(i) Geben Sie die lokale Basis zu diesem Koordinatensystem an - entweder durch explizite Rechnung, oder indem Sie die Basis aus Aufgabe 2 entsprechend anpassen.

All that needs to be done here is to replace the angle ψ with the new angle $\psi - \omega t$. The reason behind is that the the basis vectors,

$$\frac{\partial \boldsymbol{s}}{\partial y_i}$$

involve only spatial derivatives but not time derivatives.

(ii) Bestimmen Sie die Metrik sowie die Christoffelsymbole in diesen Koordinaten. Auch diese können Sie durch Nachdenken aus Ihren Ergebnissen aus Aufgabe 2 ableiten, d.h. es ist keine explizite Rechnung notwendig.

The metric is obtained from the scalar products of the basis vectors of the rotating coordinate system. As these basis vectors are identical up to the replacement $\psi \to \psi - \omega t$, the metric in terms of the rotating coordinate system is identical to the one of the spherical coordinates, up to this replacement. Also the Christoffel symbols are identical up to this replacement, as they are obtaind from the spatial derivatives of the metric.

(iii) Bestimmen Sie die sich aus dieser Rotation ergebende Kraft, die auf eine Punktmasse auf der Bahn $(r(t), \theta(t), \varphi(t))$ wirkt. Dies ist die Ihnen aus der Experimentalphysik vielleicht schon bekannte *Corioliskraft*.

Firstly we take what we have learned and formulated about fictitious force in question 2(iv). Then we substitute with,

$$\dot{\theta} = 0, \qquad \dot{r} = 0, \qquad \dot{\psi}(t) = -\omega$$

then we have,

$$\frac{F^r(\gamma(t))}{m} = -r\omega^2\sin^2\theta \qquad \frac{F^\theta(\gamma(t))}{m} = -\omega^2\sin\theta\cos\theta, \qquad \frac{F^\psi(\gamma(t))}{m} = 0$$