Exercises for Quantum Field Theory (TVI/TMP)

Problem Set 6

1 BV quantization - photon

Consider the electromagnetic action

$$S_0 = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1}$$

We want to derive the BV extended action of this.

(i) Since this action has a gauge freedom, given by $\delta A_\mu = \partial_\mu \alpha$, we need to introduce a ghost field η to account for that. It is odd and of ghost number $gh(\eta) = 1$. To those fields, we add anti-fields $\bar{A}^\mu, \bar{\eta}$. Using the general formula $gh(\bar{\phi}_a) = -gh(\phi^a) - 1$, calculate the ghost numbers of these fields. Which fields are even, which are odd?

(ii) We define a Poisson bracket

$$\{F, G\} = \int \mathrm{d}^4x \left(\frac{\delta_R F}{\delta \bar{\phi}_a(x)} \frac{\delta_L G}{\delta \phi^a(x)} \right) - \left(\frac{\delta_R F}{\delta \phi^a(x)} \frac{\delta_L G}{\delta \bar{\phi}_a(x)} \right) \tag{2}$$

$$= \int \mathrm{d}^4x \left(\frac{\delta_R F}{\delta \bar{A}_\mu(x)} \frac{\delta_L G}{\delta A^\mu(x)} - \frac{\delta_R F}{\delta A^\mu(x)} \frac{\delta_L G}{\delta \bar{A}_\mu(x)} + \frac{\delta_R F}{\delta \bar{\eta}(x)} \frac{\delta_L G}{\delta \eta(x)} - \frac{\delta_R F}{\delta \eta(x)} \frac{\delta_L G}{\delta \bar{\eta}(x)} \right). \tag{3}$$

Here, $\frac{\delta_{L/R}}{\delta \phi^a}$ stands for the derivative acting from the left, resp. from the right.

At zeroth order in the anti-fields, the BV action should is given by S_0. To first order, the extended action is defined to generate Gauge-BRST transformations of the field A^μ through the Poisson bracket, i.e.

$$\delta_S A^\mu = \{S, A^\mu(x)\} = \partial^\mu \eta. \tag{4}$$

Show that the action

$$S = S_0 - \int \mathrm{d}^4x \bar{A}_\mu \partial^\mu \eta(x) \tag{5}$$

generates this transformation.

(iii) Terms in higher anti-field number in S are chosen such that S satisfies the classical master equation,

$$\{S, S\} = 0. \tag{6}$$

Convince yourself that this equation is actually non-trivial (unlike in the case of the Hamiltonian Poisson bracket). (*Hint*: Show that $\frac{\delta_L X}{\delta \phi} = (-)^{gh(\phi)(gh(X)+1)} \frac{\delta_R X}{\delta \phi}$. Also, can you guess what the ghost number of a functional of the fields is?) Afterwards, show that (4) already does the job. It is therefore our BV extended action.

(iv) Gauge fixing in the BV formalism means that we want to set the anti-fields to specific values. It is done using a functional Ψ, the *gauge fixing fermion*, by declaring $\bar{\phi}_a = \frac{\Psi(\phi)}{\delta \phi^a}$. For this to make sense, show that $gh(\Psi) = -1$. However, $\Psi(\phi)$ should not depend on anti-fields, so at this point, it is impossible to construct such a Ψ.

To fix this, we extend our field space even more. We introduce a field η^* with $gh(\eta^*) = -1$ together with a corresponding anti-field $\bar{\eta}^*$. Wo want to include it to the action, but in a trivial way. Since S

should have ghost number zero, we introduce yet another pair (B, B^*), with $gh(B) = 0$ and couple it to $\bar{\eta}^*$, using

$$S_t = \int \mathrm{d}^4 x B(x) \bar{\eta}^*(x). \tag{7}$$

Defining $S_{tot} = S + S_t$ obviously does not change dynamics, since the equations of motion for the new fields are trivial. However, using η^*, we can now write down a Ψ of proper ghost number.

(v) Apart from the above, the only condition on Ψ is that it is such that the gauge fixed action has a propagator. Consider

$$\Psi = \int \mathrm{d}^4 x\, \eta^*(x)(-\frac{B(x)}{2\xi} + F(A, x)), \tag{8}$$

where, for example $F(A, x) = \partial_\mu A^\mu$. Write down the gauge fixed action S_Ψ, which is given by

$$S_\Psi(\phi^a) = S_{tot}(\phi^a, \bar{\phi}_a = \frac{\delta_L \Psi}{\delta \phi^a}). \tag{9}$$

You should find

$$S_\Psi = S_0 - \int \mathrm{d}^4 x\, \mathrm{d}^4 y\, \eta^*(x) \frac{\delta F(A, x)}{\delta A^\mu(y)} \partial_\mu \eta(y) - \int \mathrm{d}^4 x B(x) \left(\frac{B(x)}{2\xi} - F(A, x) \right). \tag{10}$$

When you replace B by its solution to the equation of motion, you should obtain a familiar action.

2 BV quantization - abelian 2-form theory

We give an example for a reducible theory. Consider a 2-form field A taking values in some abelian Lie algebra with field strength $F = \mathrm{d}A$. The classical action is then

$$S_0 = -\frac{1}{2} \int F \wedge *F, \tag{11}$$

where the $*$-operation is the Hodge dual. In the following we will keep using the coordinate free notation. If you prefer to work with indices you may want to use the identity $A \wedge *B = g(A, B)\mathrm{d}V$, where g is the metric and $\mathrm{d}V$ is the volume form.
Since $\mathrm{d}^2 = 0$ this action is clearly invariant under

$$\delta A = \mathrm{d}\sigma_1, \tag{12}$$

where λ_1 is some one-form. Note however that the gauge transformations are itself redundant. We have $\delta A = 0$ for $\sigma_1 = \mathrm{d}\sigma_0$. How many degrees of freedom (propagating as well as non-propagating) does this theory have? Since we have a first stage reducible theory we have to introduce ghosts C_1 and ghosts for ghosts C_0 (the labels remind us that we deal with 1- and 0-forms). Or set of fields is therefore $\phi^A = (A, C_1, C_0)$ with ghost numbers $(0, 1, 2)$.
As in the preceding exercise we extend the set of fields by a set of anti-fields $\phi^*_A = (A^*, C_1^*, C_0^*)$ of ghost number $(-1, -2, -3)$. The classical BV-action S should generate gauge transformations

$$\delta A = \mathrm{d}C_1 + ..., \quad \delta C_1 = \mathrm{d}C_0 + ..., \quad \delta C_0 = 0 + ... \,. \tag{13}$$

Show that the extended action

$$S = S_0 + S_1 = S_0 + \int A^* \wedge *\mathrm{d}C_1 + \int C_1^* \wedge *\mathrm{d}C_0 \tag{14}$$

generates (13) and already solves the classical master equation.
To gauge fix this action we again introduce trivial pairs. Because we gauge-fix two fields A and C_1 we need at least two trivial pairs. We fix A using (B_1, λ_1) with ghost numbers $(-1, 0)$ and C_1 using (B_0, λ_0) with ghost numbers $(-2, -1)$. The reason for the particular ghost numbers is that the B_i serve as anti-ghosts for the C_i, so they have opposite ghost numbers. The λ_i are Lagrange multipliers to fix A and C_1 and therefore have opposite ghost numbers with respect to the fields they fix. The action for the trivial pairs now reads

$$S_t = \int B_1^* \wedge *\lambda_1 + \int B_0^* \wedge *\lambda_0. \tag{15}$$

We pick the following gauge fixing fermion:

$$\Psi = \int \mathrm{d}B_1 \wedge *A + \int \mathrm{d}B_0 \wedge *C_1. \tag{16}$$

Convince yourself that it has the right properties and imposes a Lorenz gauge on both A and C_1. You should find

$$S_\Psi = -\frac{1}{2}\int F \wedge *F + \int \mathrm{d}B_1 \wedge *\mathrm{d}C_1 + \int \mathrm{d}B_0 \wedge *\mathrm{d}C_0 + \int A \wedge *\mathrm{d}\lambda_1 + \int C_1 \wedge *\mathrm{d}\lambda_0. \tag{17}$$

Unfortunately we are not done yet. The redundancy of the fields A and C_1 is fixed through a delta-function. However there is now a redundancy in the anti-ghost B_1, namely

$$\delta B_1 = \mathrm{d}\sigma_0. \tag{18}$$

So we need still another trivial pair to fix B_1. Mimic what we have done for A and C_1, i.e. introduce a trivial pair with appropriate ghost numbers, add a term to the gauge fixing fermion to eliminate the gauge degree of freedom and derive the gauge-fixed action.

3 Finite dimensional BRST-BV (last year's exam problem)

Consider the following integral

$$\langle f \rangle = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y e^{-S_0(x,y)} f(x-y), \tag{19}$$

with action $S_0(x,y) = \frac{1}{2}(x-y)^2$.

(i) (2pt) Identify the "gauge" symmetry of this action – write down explicitly the transformation(s) under which the action is invariant.

(ii) (5pt) Use the Faddeev-Popov trick to gauge fix this action, with $F(x,y) = x + y + G(x-y)$ as a gauge fixing condition, where $G(z)$, $z \in \mathbb{R}$, is some arbitrary regular function. You should find

$$S_0 + S_{gh} = \frac{1}{2}(x-y)^2 + C\eta^*\eta, \tag{20}$$

with some (specific) constant C, and (η^*, η) a pair of fermionic variables. The complete gauge fixed integral should be

$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}\bar{\eta}\mathrm{d}\eta f(x-y) e^{-S_0 - S_{gh}} \delta(F(x,y)) \tag{21}$$

(iii) (2pt) Average the Gauge fixing condition $F(x,y) = c$ over c with the weight function $e^{-\frac{1}{2}c^2}$. You should find

$$S_{tot} = \frac{1}{2}(x-y)^2 + C\eta^*\eta + \frac{1}{2}F^2. \tag{22}$$

(iv) (4pt) Write down the concrete expressions of the BRST transformations $\delta_B \phi = \zeta \mathfrak{s}(\phi)$ for all the four variables, $\phi = (x, y, \eta, \eta^*)$. Show that S_{tot} is invariant under them.

(v) (3pt) The BRST transformations do not yet square to zero. Use

$$\int e^{-\frac{1}{2}b^2 + ibF} db \sim e^{-\frac{1}{2}F^2} \tag{23}$$

to integrate in a Lagrange multiplier b and write down the new BRST transformations. Show that $\delta_B^2 = 0$.

(vi) (3pt) Go back to the original S_0. Find its (minimal) BV extended action and show that it satisfies the classical master equation (without using the equations of motion).