Exercises for Quantum Field Theory (TVI/TMP)

Problem Set 5

1 Two-point function of the Dirac current

Consider the free Dirac fermion described by the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi. \tag{1}$$

(i) Argue using the fermionic Gaussian path integral that the propagator for Dirac field is

$$S_F(x-y) \equiv \langle 0| T\psi_j(x)\bar{\psi}_k(y) |0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{2}$$

Note: we could find the propagator also using the canonical quantization for Dirac field as two-point function $\langle 0| T\psi(x)\overline{\psi}(y) |0\rangle$. The time-ordering for fermions has an additional minus sign if we exchange the fermionic fields.

(ii) Calculate using Wick theorem (in the free theory) the two-point function

$$\langle 0| j_\mu(x) j_\nu(y) |0\rangle \tag{3}$$

of Dirac current

$$j^\mu(x) =: \overline{\psi}(x)\gamma^\mu\psi(x) : \tag{4}$$

where the normal ordering : . . . : means that we do not consider the internal Wick contractions inside of $j^\mu(x)$. Do not try to evaluate any of the momentum integrals.

(iii) Take the Fourier transform of the two-point function,

$$\int \mathrm{d}^4x\mathrm{d}^4y e^{ikx+ily} \langle 0| j_\mu(x) j_\nu(y) |0\rangle \tag{5}$$

and show that it is equal to

$$(2\pi)^4\delta^4(k-l)(-1)\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{Tr}(\gamma^\mu i(k\!\!\!/ + p\!\!\!/ + m)\gamma^\nu i(p\!\!\!/ + m))}{(p^2 - m^2 + i\epsilon)((k+p)^2 - m^2 + i\epsilon)}. \tag{6}$$

2 Photon self-energy using dimensional regularization

We want to evaluate the one-loop Feynman diagram

$$i\Pi^{\mu\nu}(q) = (-ie)^2(-1)\int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\left(\gamma^\mu \frac{i}{k\!\!\!/ - m + i\epsilon}\gamma^\nu \frac{i}{k\!\!\!/ + q\!\!\!/ - m + i\epsilon}\right) \tag{7}$$

in dimensional regularization. We are in particular interested in this quantity for $d = 2$ ($1 + 1$ dimensional QED) and $d = 4$ ($3 + 1$ dimensional QED). [Some of the formulas below are specialized to $d = 2$ but you can try to stay general as long as you can, because also the $4d$ case has important applications.]

(i) Which correlation function in QED can have one-loop contribution of this form? What is the corresponding Feynman diagram?

(ii) First of all, multiply the denominator factors $\not{k} - m + i\epsilon$ by their conjugates to bring all the gamma matrix algebra to numerator. Next derive and use the gamma matrix identities for $\mathrm{Tr}(\gamma^\mu\gamma^\nu)$ and $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ to eliminate the gamma matrices completely.

(iii) Now combine the bosonic propagators in the denominator using the Feynman parameters

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}. \tag{8}$$

Shift the integration momentum $k^\mu \to l^\mu \equiv k^\mu + (\cdots)q^\mu$ to have only l^2 and no mixed terms $l \cdot a$ in the denominator. Finally the odd powers of integration momentum l^μ in the denominator will drop out when integrated over l by symmetry. You should arrive at something equivalent to

$$i\Pi^{\mu\nu}(q) = -2e^2 \int \frac{d^2 l}{(2\pi)^2} \int_0^1 dx \frac{2l^\mu l^\nu - \eta^{\mu\nu} l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2)}{(l^2 + x(1-x)q^2 - m^2)^2} \tag{9}$$

(iv) By counting powers of l, this expression diverges logarithmically at large l. We thus use so called dimensional regularization to calculate this integral. Pratically, what this means is that we will evaluate this integral in d dimensions where d is considered as a complex parameter. To do so, we first use the symmetry to replace $l^\mu l^\nu \to \frac{1}{d} l^2 \eta^{\mu\nu}$ (why?) and then perform a Wick rotation, $l^0 = i l_E^0$.

(v) Next we need to integrate the scalar integrals

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{2-d/2}} \tag{10}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{\Delta^{1-d/2}} \tag{11}$$

If you want, derive these formulas. Even if you don't want, use them to evaluate the integral. It turns out that although the term with l_E^2 in the numerator was logarithmically divergent, after calculating it in d dimensions the result has finite limit as $d \to 2$, concretely

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2 \eta^{\mu\nu} - \frac{2}{d} l_E^2 \eta^{\mu\nu}}{(l_E^2 + \Delta)^2} = -\frac{\eta^{\mu\nu}}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{1-d/2}} \overset{d \to 2}{\to} \frac{-\eta^{\mu\nu}}{4\pi}. \tag{12}$$

(vi) The final expression that you get should be

$$i\Pi^{\mu\nu}(q) = \frac{-ie^2}{\pi} \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 \frac{x(1-x)dx}{m^2 - x(1-x)q^2}. \tag{13}$$

We can now consider the massless limit and find the final answer

$$i\Pi^{\mu\nu}(q) = \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) \frac{ie^2}{\pi q^2} \equiv \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) i\Pi(q). \tag{14}$$

(vii) The previous result can be interpreted as a generation of photon mass by fermionic loop corrections. Consider higher order corrections to photon two point-function whose Feynman diagrams are chains of alternating photon propagators

$$\frac{-i\eta^{\mu\nu}}{q^2} \tag{15}$$

and fermionic loops calculated above

$$i\Pi^{\mu\nu}(q) = \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) i\Pi(q). \tag{16}$$

The total contribution is geometric series which can be resummed. As result you should find

$$\frac{-i\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2(1 - \Pi(q^2))} - i\frac{\frac{q^\mu q^\nu}{q^2}}{q^2}. \tag{17}$$

If $\Pi(q^2)$ is regular at $q^2 = 0$ (which would be the case in $d > 2$), the quantum corrected photon propagator still has pole at $q^2 = 0$ which signifies that there is no photon mass generated by the loop corrections. On the other hand, we saw above that in $1+1$ dimensions $\Pi(q^2)$ has a pole at $q^2 = 0$. Compare the quantum corrected result with the usual propagator and determine the new photon mass.

3 BRST symmetry

(i) In the BRST formalism, we compute expectation values by the following path integral

$$\langle h(A)\rangle = \int \mathcal{D}A\mathcal{D}H\mathcal{D}\bar{H}h(A)e^{i(S[A]+\int \mathrm{d}^4x\mathrm{d}^4y\bar{H}_a(x)M^a_b(x,y)A^b(y)+\frac{\lambda}{2}\int \mathrm{d}^4xF^a(A(x))F^a(A(x)))}, \tag{18}$$

where we defined $M^a_b(x,y) = \frac{\delta F^a(A(x))}{\delta\alpha^b(y)}$. Show that the action in (18) is invariant under

$$\delta A^\mu_a = \delta\zeta D^\mu_{ab}H^b =: \delta\zeta(sA)^\mu_a, \tag{19}$$

$$\delta\bar{H}_a = -\delta\zeta\lambda F_a =: \delta\zeta(s\bar{H})_a, \tag{20}$$

$$\delta H^a = \delta\zeta\frac{1}{2}C^a{}_{bc}H^bH^c =: \delta\zeta(sH)^a. \tag{21}$$

The parameter $\delta\zeta$ is odd. This ensures that the above transformations preserve statistics.

(ii) Let us look at the term $\mathcal{L}_{gf} = \frac{\lambda}{2}F^a(A)F^a(A)$ in the Lagrangian (18). Let us introduce an auxiliary field b^a. Show that the replacement $\mathcal{L}_{gf} \mapsto \mathcal{L}'_{gf}$, where

$$\mathcal{L}'_{gf} = -\frac{1}{2\lambda}b^ab^a - b^aF^a, \tag{22}$$

leads to an equivalent path integral (Hint: Assume $\mathcal{L}'_{gf}$ and integrate out b). The new action is now invariant under

$$(sA)^\mu_a = \delta\zeta D^\mu_{ab}H^b, \tag{23}$$

$$(s\bar{H})_a = b_a, \tag{24}$$

$$(sH)^a = \frac{1}{2}C^a{}_{bc}H^bH^c, \tag{25}$$

$$(sb)^a = 0. \tag{26}$$

Show that $s^2 = 0$.