Exercises for Quantum Field Theory (TVI/TMP)

Problem Set 4

1 Feyman rules for Yang-Mills

Derive the momentum space Feynman rules for Yang-Mills theory with gauge fixing term $\mathcal{L}_{gf} = -\frac{1}{2\gamma}(\partial A)^2$ (i.e. the gluon propagator, three- and four-gluon interaction, the ghost propagator and the coupling between ghosts and gluons).

2 Faddeev-Popov Gauge Fixing

Consider the electromagnetic Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) Show that $F_{\mu\nu} = 0$ for $A_\mu = \partial_\mu \lambda$. This implies, that the path integral

$$\int \mathcal{D}A\, e^{iS[A]} f[A] \tag{2}$$

over gauge invariant functionals f (invariant under $A_\mu \to A_\mu + \partial_\mu \lambda$) is expected to give infinite results. In perturbation theory, this manifests itself in the kinetic operator, $p^\mu p^\nu - g^{\mu\nu}p^2$, not being invertible.

(ii) The process to cure this problem is called gauge fixing, as you undoubtedly know by now. We want to restrict to a certain subclass of the A, given for example by an equation $G(A) = 0$, so that each of the families $A_\mu + \partial_\mu \lambda$ (A_μ fixed, λ variable) has a unique representative in this subclass. In (2), we then restrict our integral to this subclass. We do this by factorizing the measure

$$\mathcal{D}A = J\,\mathcal{D}\lambda\,\mathcal{D}A\,\delta(G(A)), \tag{3}$$

and then drop the integration over the gauge parameter λ.

A popular choice is $G(A) = \partial_\mu A^\mu$. We define a quantity Δ implicitly via

$$1 = \Delta[A_\mu] \int \mathcal{D}\lambda\, \delta(\partial_\mu(A^\mu + \partial^\mu \lambda)). \tag{4}$$

and insert it into (2), so we have

$$\int \mathcal{D}\lambda \int \mathcal{D}A\, \Delta[A_\mu] e^{iS[A]} f[A] \delta(\partial_\mu(A^\mu + \partial^\mu \lambda)). \tag{5}$$

Show that the integrand under $\int \mathcal{D}\lambda$ in (5) is actually independent of λ. For this, assume that $\mathcal{D}\lambda$ and $\mathcal{D}A$ are invariant under shifts in λ. Then show that $\Delta(A)$ is gauge invariant, and, using this, that you can make the integrand independent of λ by a shift in the A integration variable. The upshot is, that we can now safely drop the λ integration.

(iii) It remains to determine $\Delta(A)$. For this, we formally write

$$\Delta^{-1}(A) = \int \mathcal{D}\lambda\, \delta(\partial_\mu(A^\mu + \partial^\mu \lambda)) = \int \mathcal{D}G \det \left(\frac{\delta G(A_\mu + \partial_\mu \lambda)}{\delta \lambda} \right)^{-1} \delta(G), \tag{6}$$

as we would do for ordinary integrals (we have basically done this in evaluating $\int_0^\infty \mathrm{d}p^0 \delta(p^2 - m^2)$ on the last exercise sheet). Hence

$$\Delta(A) = \det \left(\frac{\delta G(A_\mu + \partial_\mu \lambda)}{\delta \lambda} \right)\Bigg|_{G=0}. \tag{7}$$

Since $\Delta(A)$ is gauge invariant, we can choose an A, such that $F = 0$, so we can write

$$\Delta(A) = \det \left(\frac{\delta G(A_\mu + \partial_\mu \lambda)}{\delta \lambda} \right)\Bigg|_{\lambda=0}, \quad G = 0. \tag{8}$$

Expand

$$G(A_\mu(x) + \partial_\mu \lambda(x)) = G(A_\mu(x)) + \int \mathrm{d}^4 y M(A_\mu; x, y) \lambda(y), \tag{9}$$

and determine $M(A_\mu; x, y)$. So

$$\Delta(A) = \det M(A). \tag{10}$$

You should find that $\det M(A)$ is actually independent of A and therefore does not contribute to the path integral (5), up to a normalization. This is because A transforms linearly in λ. Do you know a theory where this is not the case?

(iv) We see that we can simply set $\partial_\mu A^\mu = 0$. Write

$$A^\mu = (\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box}) A_\nu + \frac{\partial^\mu \partial^\nu}{\Box} A_\nu =: A_T^\mu + A_L^\mu, \tag{11}$$

and show that this implies (with suitable boundary condition) $A_L = 0$. Is the kinetic operator invertible on the field A_T?

(v) In principle we can also consider the condition $\partial_\mu A^\mu = C(x)$. with where C is some arbitrary function. Because of gauge invariance,

$$\int \mathcal{D}A \det M(A) \delta(\partial_\mu A^\mu - C) f(A) e^{iS} \tag{12}$$

is independent of C. This implies that we can average it over C with some weight function $W(C)$, and change only its normalization. A common choice is

$$W(C) = \exp\left[-i \frac{\xi}{2} \int \mathrm{d}^4 x\, C^2(x) \right]. \tag{13}$$

Do the C integral to show that we effectively obtain the Lagrangian

$$\mathcal{L}_\xi = -\frac{1}{4} F^2 - \frac{\xi}{2} (\partial_\mu A^\mu)^2. \tag{14}$$

What is the kinetic operator of this Lagrangian? Show that it that it is invertible for all $\xi \neq 0$, and determine its inverse. Note that the above discussion shows that gauge invariant amplitudes will nevertheless be independent of ξ.