Prof. Ivo Sachs Summer term 2020

Exercises for Quantum Field Theory (TVI/TMP)

Problem set 2

Lie algebras, Classical Yang-Mills

1 Lie algebras

(i) What are the Lie algebras $\mathfrak{u}(N)$, $\mathfrak{su}(N)$, $\mathfrak{o}(N)$, $\mathfrak{sl}(N,\mathbb{R})$, $\mathfrak{gl}(N,\mathbb{C})$ of Lie groups $U(N)$, $SU(N)$, $O(N)$, $SL(N,\mathbb{R})$, $GL(N,\mathbb{C})$? What are their dimensions?

(ii) Choose a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Consider now the 3-dimensional adjoint representation and express the generators H, E and F explicitly as 3×3 matrices. Verify that the commutation relations are the same as before.

(iii) Show that the bilinear form given by a trace in the fundamental representation

$$B_{fund}(X,Y) = \mathrm{Tr}_{fund}(XY) \tag{2}$$

is invariant in the sense that

$$B([Z,X],Y) + B(X,[Z,Y]) = 0. \tag{3}$$

for any elements X, Y and Z of the Lie algebra $\mathfrak{sl}(N,\mathbb{C})$. In the case of $\mathfrak{sl}(2,\mathbb{C})$ evaluate the components of B_{fund} in the basis E, H, F.

(iv) In the case of Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ evaluate explicitly the components of the the Killing form

$$B_K(X,Y) = \mathrm{Tr}_{adj}\, \mathrm{ad}_X\, \mathrm{ad}_Y \tag{4}$$

(evaluated this time in the adjoint representation). Show that the Killing form is invariant. Here the operator ad_X in adjoint representation acts via Lie brackets,

$$\mathrm{ad}_X\, Y \equiv [X,Y]\,. \tag{5}$$

(v) In simple Lie algebra like $\mathfrak{sl}(2,\mathbb{C})$ all the invariant forms are proportional to each other. What is the relative normalization between the two invariant forms that we introduced in the case of $\mathfrak{sl}(2,\mathbb{C})$?

(vi) Consider a general Lie algebra with commutation relations

$$[T_a, T_b] = iC^c{}_{ab}T_c. \tag{6}$$

where T_a form a basis of $\mathfrak{g}$ as a vector space. Write the matrix elements of the Killing form $B(T_a,T_b)$ in terms of the structure constants $C^c{}_{ab}$.

(vii) (*) Use two times the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{7}$$

to show that the Killing form is always invariant.

2 Gauge fields, curvature

(i) Consider a set of fields ψ transforming under gauge transformations as

$$\psi(x) \mapsto \psi'(x) = U(x)\psi(x) = e^{-i\theta^a(x)T_a}\psi(x) \tag{8}$$

where T_a are some matrices satisfying the commutation relations $[T_a, T_b] = iC^c_{ab}T_c$. Let us introduce a covariant derivative

$$D_\mu\psi = \partial_\mu\psi + igA^a_\mu T_a\psi \equiv \partial_\mu\psi + igA_\mu\psi. \tag{9}$$

Find the transformation law for the gauge fields A^a_μ such that $D_\mu\psi$ transforms under the gauge transformations in the same way as ψ.

(ii) Show that under infinitesimal gauge transformation we have

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu\theta^a + C^a_{bc}\theta^b A^c_\mu. \tag{10}$$

(iii) Define the curvature tensor $G^a_{\mu\nu}$ by

$$[D_\mu, D_\nu]\,\psi = igG^a_{\mu\nu}T_a\psi \equiv igG_{\mu\nu}\psi. \tag{11}$$

Express the matrices $G_{\mu\nu}$ in terms of A_μ and the components $G^a_{\mu\nu}$ in terms of A^a_μ.

(iv) How do the quantities $G_{\mu\nu}$ transform under gauge transformations? How do $G^a_{\mu\nu}$ transform under infinitesimal gauge transformations?

3 Yang-Mills action and equations of motion

(i) Show that the Lagrangian (QCD)

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,[G_{\mu\nu}G^{\mu\nu}] + i\bar{\psi}_j(\not{D}\psi)_j - m\bar{\psi}_j\psi_j \tag{12}$$

is invariant under local gauge transformations. Here the fields ψ form a vector in fundamental N-dimensional representation of $SU(N)$ and the invariant form is normalized such that $\mathrm{Tr}\,T_aT_b = \frac{1}{2}\delta_{ab}$.

(ii) Find the Euler-Lagrange equations of motion.

(iii) Consider now the pure Yang-Mills action, i.e.

$$\mathcal{L} = -\frac{1}{4}\left[G^a_{\mu\nu}G^{\mu\nu}_a\right]. \tag{13}$$

(where due to our normalization of the gauge fields we use the metric δ_{ab} to raise and lower the indices in the adjoint representation). It is in particular invariant under the global transformations. Find the corresponding Noether currents.

(iv) Use the equations of motion to show that these currents are conserved.