Exercises for Quantum Field Theory (TVI/TMP)

Problem set 1 - Solution

1 Gamma matrices

(i) This is a straightfoward exercise with matrix multiplication. It is useful to remember that the block matrices can be multiplied just as the usual matrices but we must be careful about the order of the matrix elements. Choosing $\mu = 0 = \nu$ the Clifford algebra relation is

$$\{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = 2\eta^{00}\mathbb{1} = 2 \cdot \mathbb{1} \tag{1}$$

so we should simply verify that $(\gamma^0)^2 = \mathbb{1}$. With the explicit expression for γ^0 we have

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1} \tag{2}$$

so the first Clifford algebra relation is indeed satisfied. For $\mu = 0$ and $\nu = j$ we have

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_j + \sigma_j & 0 \\ 0 & \sigma_j - \sigma_j \end{pmatrix} = 0 \tag{3}$$

which are another three relations of the Clifford algebra. Finally, for $\mu = j$ and $\nu = k$ we have

$$\gamma^j\gamma^k + \gamma^k\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_j\sigma_k - \sigma_k\sigma_j & 0 \\ 0 & -\sigma_j\sigma_k - \sigma_k\sigma_j \end{pmatrix} \tag{4}$$

Pauli's σ-matrices satisfy the useful identity

$$\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l + \delta_{jk}\mathbb{1} \tag{5}$$

so in particular $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{1}$ and we find

$$\gamma^j\gamma^k + \gamma^k\gamma^j = -2\delta_{jk} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = 2\eta^{jk}\mathbb{1} \tag{6}$$

so we see that the remaining Clifford algebra relations are indeed satisfied.

(ii) The defining relations of the Clifford algebra can be understood in the following way: for $\mu = \nu$ the relation tells us that the square of γ^μ is ± 1 depending on the signature of the metric: $+1$ for $\mu = 0$ and -1 for $\mu = 1, 2, 3$. On the other hand, if $\mu \neq \nu$, the right-hand side is zero so γ^μ and γ^ν anticommute, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$.

Consider now an arbitrary string of γ matrices

$$\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_k}. \tag{7}$$

Let's look first at all the γ matrices with $\mu_j = 0$ (if there are any). Using the commutation relations we can move all of these to the right, picking a minus sign everytime we encounter γ^μ with $\mu \neq 0$. Once they are all on the right, we can use the relation $(\gamma^0)^2 = \mathbb{1}$ to eliminate all of them except for at most one. Now we can repeat the procedure with γ^1 etc. In the end, we see that an arbitary product of γ matrices can be reduced to a product of at most 4 matrices $\gamma^0, \gamma^1, \gamma^2$ and γ^3 with each of the gamma matrices appearing either once or not appearing at all. The total number of these products is $2^4 = 16$ which is indeed the dimension of the space of 4×4 matrices.

Looking at the list in the exercise, each line represents all the products with 0, 1, 2 etc. matices: the product of no gamma matrices is just the identity matrix. Next there are 4 first powers of gamma matrices. If we have a product of two gamma matrices γ^μ and γ^ν with $\mu \neq \nu$, we can always antisymmetrize it:

$$\gamma^\mu\gamma^\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) + \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) + \eta^{\mu\nu}\mathbb{1} \overset{\mu\neq\nu}{=} \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \equiv \gamma^{\mu\nu}. \quad (8)$$

This are exactly the matrices given on the third line and there are $\binom{4}{2}$ of them, i.e. 6. One can proceed similarly with products of 3 and 4 different gamma matrices and one finds exactly the matrices given on the last two lines of the list. In total we have

$$1 + 4 + 6 + 4 + 1 = 16 \quad (9)$$

matrices as we wanted to see. Note: the linear independence can be checked either directly or by using the trace formulas (exercise: check that all products of γ matrices listed above are traceless except for $\mathbb{1}$).

(iii) The matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the last matrix of the list (up to an overall factor). Let us have a look at the expression

$$-\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \quad (10)$$

This expression has $4! = 24$ non-zero terms, one for each permutation of $(0, 1, 2, 3)$. Exchanging any two indices of $\epsilon^{\mu\nu\rho\sigma}$ changes the sign. As discussed before, the same is true for the product $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$ if all four indices (μ, ν, ρ, σ) are different which they are. The two signs therefore cancel and we find 24 times the same term,

$$-\frac{i}{4!} \times 4! \times \epsilon^{0123}\gamma_0\gamma_1\gamma_2\gamma_3 = +i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (11)$$

which is what we wanted to show. The anticommutativity is easy: we have

$$\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = (-1)^3 i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5. \quad (12)$$

When moving γ^μ to the left, we encountered three times γ^ν with $\nu \neq \mu$ which produced three minus signs and once γ^μ itself which does not produce any additional sign. To find the square of γ^5, we write

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2 = +\mathbb{1}. \quad (13)$$

We first moved γ^0 to the left, exchanging it with γ^3, γ^2 and γ^1, producing three signs. Afterwards γ^1 produced two signs and γ^2 one sign. In the last step we used the formula $(\gamma^\mu)^2 = \pm 1$ (three times minus and once plus).

(iv) We see from our explicit matrix representation that indeed $(\gamma^0)^\dagger = \gamma^0$ while $(\gamma^j)^\dagger = -\gamma^j$ (where we used the hermiticity of Pauli's σ matrices). But we also know from the Clifford algebra relations that γ^0 commuties with self while it anticommutes with γ^j, $j = 1, 2, 3$. Combining these two facts, we indeed see that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu. \quad (14)$$

We also see easily that γ^5 is hermitian:

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = -i\gamma^0(\gamma^3\gamma^2\gamma^1\gamma^0)\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (15)$$

This also shows (with $+ - --$ metric) that we cannot find a representation of γ^μ such that γ^j are hermitian: the Clifford algebra relations imply that $(\gamma^j)^2 = -\mathbb{1}$ which is never satisfied by a hermitian matrix. Any hermitian matrix has real eigenvalues so its square needs to have non-negative eigenvalues which is obviously not true for $-\mathbb{1}$.

2 Dirac equation

(i) Writing down indices of vectors and matrices explicitly, the Dirac equation looks like

$$i(\gamma^\mu)^j{}_k\partial_\mu\psi^k(x) - m\delta^j_k\psi^k(x) = 0. \quad (16)$$

where $j, k = 1, \ldots, 4$ are the spinor indices where the gamma matrices act. These are quite different from the vector index $\mu = 0, \ldots, 3$. For example, in d dimensional space-time with d even the vector index μ would have values $\mu = 0, \ldots, d-1$ while the space of Dirac spinors is complex $2^{\frac{d}{2}}$-dimensional. For $d = 3+1$ these two dimensions agree.

Following the hint, we multiply the Dirac equation on the left by the conjugate operator $(-i\gamma^\nu\partial_\nu - m)$. We find

$$0 = (-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi = \left(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - \cancel{im\gamma^\mu\partial_\mu} + \cancel{im\gamma^\nu\partial_\nu} - m^2\right)\psi \tag{17}$$

$$\overset{sym}{=} \left(\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\mu\partial_\nu - m^2\right)\psi \tag{18}$$

$$\overset{Clif}{=} \left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\right)\psi \tag{19}$$

We first symmetrized the product of gamma matrices over μ and ν because it is contracted with a symmetric object $\partial_\mu\partial_\nu$. In the following step we used the Clifford algebra relations. The resulting equation does not have any gamma matrices anymore, so the scalar operator $(\partial^2 - m^2)$ has to kill each component of ψ independently, i.e. each component of ψ satisfies the Klein-Gordon equation. Klein-Gordon equation expresses the usual relation between energy and mass ($E^2 = m^2 + \vec{p}^2$) via the identification $p_\mu = i\partial_\mu$.

(ii) The conjugate spinor $\bar{\psi}$ is defined as

$$\bar{\psi} = \psi^\dagger\gamma^0. \tag{20}$$

where † denotes the hermitian conjugate, i.e. complex conjugate and transpose. The hermitian conjugate of the Dirac equation is

$$0 = \psi^\dagger\left(-i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu - m\right) \equiv -i\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger - m\psi^\dagger \tag{21}$$

Note that hermitian conjugation reverses the order of the gamma matrix and the Dirac spinor ψ so if we want to keep ∂_μ and γ^μ together we need to differentiate on the left which is denoted by the arrow.

To have an equation for $\bar{\psi}$ and to get rid of the hermitian conjugates of the gamma matrices we multiply the equation from the right by γ^0:

$$0 = \psi^\dagger\left(-i(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu - m\right)\gamma^0 = \psi^\dagger\gamma^0\left(-i\gamma^\mu\overleftarrow{\partial}_\mu - m\right) \equiv \bar{\psi}\left(-i\gamma^\mu\overleftarrow{\partial}_\mu - m\right) \tag{22}$$

where we used the identity from the first exercise.

(iii) The least action principle states that the classical equations of motion are obtained as extrema of the classical action

$$S = \int d^4x\mathcal{L}(\psi(x), \bar{\psi}(x)). \tag{23}$$

To find the extrema, we vary the action with respect to ψ and $\bar{\psi}$ and require the variation of the action to vanish:

$$0 = \delta S = \int d^4x\left[\delta\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\delta\psi\right]. \tag{24}$$

In the second term we can integrate by parts, neglecting the boundary term (assuming for example that the variations vanish at the boundary – we can always restict to such variations). We find

$$0 = \delta S = \int d^4x\left[\delta\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\psi}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m)\delta\psi\right]. \tag{25}$$

If we have a real action depending on a complex quantity, we can think of the complex quantity and its complex conjugate as being two independent quantities. This means that we immediately find the Dirac equation and its conjugate as classical equations of motion:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \bar{\psi}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0. \tag{26}$$

which is exactly the Dirac equation and its conjugate.

Independence of ψ and $\bar{\psi}$ If you don't know that you should treat ψ and $\bar{\psi}$ as independent, you can always do a honest calculation with real quantities: the real part and the imaginary part of ψ are clearly independent. Let us write abstractly

$$0 = \delta S[\phi, \bar{\phi}] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}(\delta\phi_R + i\delta\phi_I) + \frac{\partial \mathcal{L}}{\partial \bar{\phi}}(\delta\phi_R - i\delta\phi_I) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi_R + i\delta\phi_I) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})}\partial_\mu(\delta\phi_R - i\delta\phi_I) \right] \tag{27}$$

so the Euler-Lagrange equations are

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\phi}} \tag{28}$$

from the variation with respect to ϕ_R and

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) + \frac{\partial \mathcal{L}}{\partial \bar{\phi}} \tag{29}$$

from the variation with respect to ϕ_I. We see that the sum and the difference of these equations is

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \tag{30}$$

and

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\phi}} \tag{31}$$

which are exactly the Euler-Lagrange equations that we would get if we treat ϕ and $\bar{\phi}$ as independent.

3 Noether currents for Dirac equation

(i) In this problem we are assuming that there exists a certain symmetry transformation acting on the fields ϕ_j and their derivatives and such that the action does not change. At the level of the Lagrangian density it is enough if under such transformation it changes by a total derivative: using the usual assumption about boundary terms changing the Lagrangian density by a total derivative does not change the equations of motion. Usual space-time symmetries are exactly of this kind, they don't leave the Lagrangian density invariant but instead change it by a total derivative.

For the derivation of Noether's theorem we also need to assume that we have a continuous group of symmetry transformations, in particular we need to be able to do infinitesimally small symmetry transformations. This being said, let us assume that we have a symmetry such that the Lagrangian density changes as

$$\mathcal{L}(\phi_j + \epsilon\delta\phi_j, \partial(\phi_j + \epsilon\delta\phi_j)) = \mathcal{L}(\phi_j, \partial\phi_j) + \epsilon\partial_\mu K^\mu(\phi_j, \partial\phi_j) + \mathcal{O}(\epsilon^2). \tag{32}$$

where ϵ is an infinitesimal parameter. We want to see that the current

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}\delta\phi_j - K^\mu \tag{33}$$

is conserved if the equations of motion are satisfied (this assumption is important!). The equations of motion derived from the action with Lagrangian density $\mathcal{L}$ are the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_j} = 0. \tag{34}$$

The conservation of the current J^μ is the continuity equation

$$0 \stackrel{?}{=} \partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \right)\delta\phi_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}\partial_\mu\delta\phi_j - \partial_\mu K^\mu. \tag{35}$$

In the first term we can use the Euler-Lagrange equations and we find

$$\partial_\mu J^\mu = \frac{\partial \mathcal{L}}{\partial \phi_j}\delta\phi_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}\partial_\mu\delta\phi_j - \partial_\mu K^\mu. \tag{36}$$

But the right-hand side vanishes by our assumption on invariance of $\mathcal{L}$: these terms are exactly $\mathcal{O}(\epsilon)$ terms of (32). This finishes the proof of Noether's theorem.

(ii) We now apply the previous discussion to Dirac's equation. It is easy to see that the Lagrangian is invariant under the transformation

$$\psi(x) \to e^{-iq\alpha}\psi(x) \tag{37}$$

where q is the electric charge of the particle and α is the parameter of the $U(1)$ symmetry trasformation. So far only their product $q\alpha$ enters but if we had particles with different electric charge, the difference between these would be important. Choosing α to be small ($\alpha \sim \epsilon$), we can read off the infinitesimal transformation of ψ and $\bar{\psi}$,

$$\delta\psi = -iq\psi, \qquad \delta\bar{\psi} = +iq\bar{\psi}. \tag{38}$$

The Lagrangian is invariant on nose so $K^\mu \equiv 0$. The conserved current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi} \tag{39}$$

$$= \bar{\psi}i\gamma^\mu(-iq\psi) = q\bar{\psi}\gamma^\mu\psi. \tag{40}$$

We treat ψ and $\bar{\psi}$ as independent fields. We could again split them to real and imaginary parts as before and we would find the same answer. We also used the matrix notation. For those who are not comfortable with it yet, you can just write all the equations in the components and again you should find the same result. Not that it is convenient to use the following rule: whenever I remove any field from the Lagrangian by taking the partial derivative, I should replace it by its variation at exactly the same place including all the index structure.

Note also that since our form of the Lagrangian didn't have any derivatives of $\bar{\psi}$, the conjugate terms in the formula for the Noether current didn't contribute.

Finally, we should verify the conservation of the current. We have

$$\partial_\mu J^\mu = q(\partial_\mu\bar{\psi})\gamma^\mu\psi + q\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{41}$$

$$\overset{Dirac}{=} imq\bar{\psi}\psi - imq\bar{\psi}\psi = 0. \tag{42}$$

where we used the Dirac equation in the form

$$\gamma^\mu\partial_\mu\psi = -im\psi, \qquad \partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi}. \tag{43}$$

4 Local symmetries and QED

(i) We saw that Dirac's equation was invariant under $U(1)$ transformations

$$\psi(x) \to e^{-iq\alpha}\psi(x) \tag{44}$$

for α independent of the position. But it is not invariant if α depends on x:

$$(i\gamma^\mu\partial_\mu - m)e^{-iq\alpha(x)}\psi(x) = e^{-iq\alpha(x)}(i\gamma^\mu\partial_\mu - m)\psi(x) + q(\partial_\mu\alpha)\gamma^\mu e^{-iq\alpha(x)}\psi(x). \tag{45}$$

If there wasn't for the last term and if $\psi(x)$ solved the Dirac equation, also the transformed $\psi(x)$ would satisfy Dirac equation. But we can fix this problem by replacing $\partial_\mu\psi$ by

$$D_\mu\psi = (\partial_\mu + ieqA_\mu)\psi \tag{46}$$

where $A_\mu(x)$ is a new field. The constant e is introduced for future convenience (it has a physical meaning of the elementary charge). The Lagrangian density after this replacement is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - eq\gamma^\mu A_\mu - m)\psi \tag{47}$$

Let's see how this modified action transforms under the local $U(1)$ transformation if we also simultaneously transform the field A_μ as

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{48}$$

(note that this is trivial if α is constant). We have

$$\mathcal{L} \to \bar{\psi}e^{iq\alpha}(i\gamma^\mu\partial_\mu - eq\gamma^\mu A_\mu + q\gamma^\mu(\partial_\mu\alpha) - m)e^{-iq\alpha}\psi \tag{49}$$

$$= \bar{\psi}\cancel{e^{iq\alpha}}\cancel{e^{-iq\alpha}}(i\gamma^\mu\partial_\mu + \cancel{q\gamma^\mu(\partial_\mu\alpha)} - eq\gamma^\mu A_\mu - \cancel{q\gamma^\mu(\partial_\mu\alpha)} - m)\psi = \mathcal{L} \tag{50}$$

so the Lagrangian is indeed invariant under the local gauge transformations.

(ii) The covariant derivative transforms as

$$D_\mu\psi = (\partial_\mu + ieqA_\mu)\psi \to (\partial_\mu + ieqA_\mu + iq(\partial_\mu\alpha))e^{-iq\alpha}\psi = e^{-iq\alpha}(\partial_\mu + ieqA_\mu)\psi = e^{-iq\alpha}D_\mu\psi \tag{51}$$

so it transforms exactly in the same way as ψ itself. This is one of the main reasons for introducing these covariant derivatives.

(iii) For this we just write (47) as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eqA_\mu\bar{\psi}\gamma^\mu\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eA_\mu J^\mu \tag{52}$$

where J^μ is the Noether's current derived before. Note that things are not so simple in general, if we considered instead of Dirac fermion a charged scalar field satisfying the Klein-Gordon equation, there would be also terms containing $A_\mu A^\mu$ in the action!

(iv) By the definition of the curvature tensor we have

$$\begin{aligned} ieqF_{\mu\nu}(x)\psi(x) &= D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) && (53)\\ &= (\partial_\mu + ieqA_\mu)(\partial_\nu + ieqA_\nu)\psi - (\mu \leftrightarrow \nu) && (54)\\ &= \cancel{\partial_\mu\partial_\nu\psi} + ieq(\partial_\mu A_\nu)\psi + \cancel{ieqA_\nu\partial_\mu\psi} + \cancel{ieqA_\mu\partial_\nu\psi} + \cancel{(ieq)^2A_\mu A_\nu\psi} - (\mu \leftrightarrow \nu) && (55)\\ &= ieq(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi && (56) \end{aligned}$$

We can therefore identify

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{57}$$

Most terms cancelled in the antisymmetrization, in particular all the derivatives of ψ. This is an important property of $F_{\mu\nu}$. It is easy to see that $F_{\mu\nu}$ is invariant under the gauge transformations:

$$\begin{aligned} ieqF_{\mu\nu}\psi &= D_\mu D_\nu\psi - D_\nu D_\mu\psi \to D'_\mu D'_\nu\psi' && (58)\\ &= D'_\mu D'_\nu e^{-iq\alpha}\psi - (\mu \leftrightarrow \nu) && (59)\\ &= e^{-iq\alpha}D_\mu D_\nu\psi - (\mu \leftrightarrow \nu) && (60)\\ &= ieqF'_{\mu\nu}e^{-iq\alpha}\psi && (61) \end{aligned}$$

so $F'_{\mu\nu}(x) = F_{\mu\nu}(x)$. One can also do it directly from the formula (57) by plugging in the transformation of $A_\mu(x)$.

(v) The Euler-Lagrange equations for ψ and $\bar{\psi}$ as just like before and we find the charged version of Dirac's equation

$$(i\gamma^\mu D_\mu - m)\psi(x) = (i\gamma^\mu\partial_\mu - eq\gamma^\mu A_\mu - m)\psi(x) = 0 \tag{62}$$

as well as its conjugate

$$\bar{\psi}(x)(-i\gamma^\mu\overleftarrow{D}_\mu - m) = \bar{\psi}(x)(-i\gamma^\mu\overleftarrow{\partial}_\mu - eq\gamma^\mu A_\mu - m) = 0 \tag{63}$$

Note that the covariant derivative acts on $\bar{\psi}$ with the opposite charge:

$$D_\mu\bar{\psi} = \partial_\mu\psi - ieqA_\mu\bar{\psi} \tag{64}$$

which we can see also by Dirac conjugating the equation (46). To find the equations of motion for the gauge field A_μ, we first vary

$$\begin{aligned} \delta\int d^4x\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) &= -\tfrac{1}{2}\delta\int d^4x\,[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)] && (65)\\ &= -\int d^4x\,[(\delta\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\delta\partial_\mu A_\nu)(\partial^\nu A^\mu)] && (66)\\ &= -\int d^4x(\delta\partial_\mu A_\nu)\,[(\partial^\mu A^\nu) - (\partial^\nu A^\mu)] && (67)\\ &= -\int d^4x F^{\mu\nu}\delta\partial_\mu A_\nu && (68) \end{aligned}$$

so

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}. \tag{69}$$

We also have

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial(-eqA_\mu \bar{\psi}\gamma^\mu \psi)}{\partial A_\nu} = -eq\bar{\psi}\gamma^\nu \psi = -eJ^\nu \tag{70}$$

so finally the Maxwell's equations (Euler-Lagrange equations for the Lagrangian we are considering) are

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} + eJ^\nu. \tag{71}$$