Exercises for Quantum Field Theory (TVI/TMP)

Problem set 1

Dirac equation, global and local symmetries, Noether theorem

1 Gamma matrices

(i) Consider a four-tuple of 4×4 matrices γ^μ, $\mu = 0, 1, 2, 3$ satisfying the Clifford algebra relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}. \tag{1}$$

Verify that one possible explicit representation (Weyl) of these matrices is given by

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} \\ \gamma^j &= \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \end{aligned} \tag{2}$$

where σ_j are the usual 2×2 Pauli matrices, $\sigma_j\sigma_k = \delta_{jk}\mathbb{1}_{2\times2} + i\epsilon_{jkl}\sigma_l$.

Solution: We use the convention $\eta^{\mu\nu} = \text{diag}(+, -, -, -)$. We begin by checking the anti-commutator of spatial $\gamma^i, i = 1, 2, 3$. We have

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} + (i \leftrightarrow j) = \begin{pmatrix} -\{\sigma_i, \sigma_j\} & 0 \\ 0 & -\{\sigma_i, \sigma_j\} \end{pmatrix}. \tag{3}$$

The Pauli matrices are defined to satisfy the Clifford relations in three spatially flat dimensions. (by "spatial flat" we mean that we use the metric $g_{ij} = \delta_{ij}$) Hence, they satisfy the relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_{2\times2}. \tag{4}$$

The σ_i are to 3d Euclidean space what the γ^μ are to 4d Minkowski space. It immediately follows that (1) is satisfied when $i, j \in \{1, 2, 3\}$. It remains to check (1) for $(\mu, \nu) = (0, i)$ and $(\mu, \nu) = (0, 0)$. But these are trivial computations.

(ii) Argue using the defining relation (1) that the algebra of γ-matrices has a basis given by

$$\begin{gathered} \mathbb{1} \\ \gamma^\mu \\ \gamma^{\mu\nu} \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \\ \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \\ \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \end{gathered} \tag{5}$$

How many of these matrices do we have in total? Hint: they should be as many as there are linearly independent 4×4 matrices.

Solution: By "the algebra of γ-matrices", we mean the algebra generated by the γ-matrices. That is, it is obtained by adding any power (including the zeroth power) of γ-matrices. This algebra structure is what is called the Clifford algebra. We want to show that the matrices (5) form a linear independent basis of the vector space underlying this algebra. We want to do so without making reference to the explicit representation of the gamma matrices.

As a warm up, suppose that we have gamma matrices satisfying (1) with $\eta^{\mu\nu} = 0$. This means that all the γ anti-commute. In particular, $(\gamma^\mu)^2 = 0$. In this case, the Clifford algebra is just the exterior algebra with four generators. It is generated by

$$\gamma^{\mu_1} \cdot \gamma^{\mu_2} \cdots \gamma^{\mu_i}, \tag{6}$$

with $0 \leq i \leq 4$ and $\mu_i < \mu_j$ when $i < j$. There are $1+4+6+4+1 = 16$ of these. Equivalently, we could also use the antisymmetrized version of (6), since

$$\gamma^{\mu_1} \cdots \gamma^{\mu_i} = \gamma^{[\mu_1} \cdots \gamma^{\mu_i]}, \tag{7}$$

where $A^{[i_1} \cdots A^{i_n]} = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) A^{i_{\sigma(1)}} \cdots A^{i_{\sigma(n)}}$. The sum runs over all members of the permutation group S_n of n elements.

When $\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$, the argument does not really change. The algebra is still generated by (6), since any product of γ matrices $\gamma^{i_1} \cdots \gamma^{i_n}$ can be arranged such that $i_1 \leq ... \leq i_n$ up to a sign. Further, by using (1), any power $(\gamma^{\mu_k})^n$ is proportial to $\mathbb{1}$ or γ^{μ_k}, depending on whether n is even or odd.

(iii) Let us define the combination

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{8}$$

Show that ($\epsilon^{0123} = +1$)

$$\gamma^5 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \tag{9}$$

Show that γ^5 anticommutes with γ^μ and compute $(\gamma^5)^2$.

Solution: We write

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\sum_{\sigma \in S_4} \mathrm{sign}(\sigma)\gamma^{\sigma(0)}\gamma^{\sigma(1)}\gamma^{\sigma(2)}\gamma^{\sigma(3)} = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \tag{10}$$

Note that $\epsilon_{0123} = (-)^3\epsilon^{0123} = -1$. To check the anti-commuting property, suppose we are given a fixed γ^μ, $\mu = 0,1,2,3$, and we want to commute it trough γ^5. γ^μ commutes with itself and anti-commutes with all the other γ matrices. Since γ^5 contains each γ-matrix exactly once, this tells us that $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Finally, anti-commutativity of γ-matrices allows us to write $\gamma^5 = i\gamma^3\gamma^2\gamma^1\gamma^0$. Hence,

$$\gamma^5\gamma^5 = (i)^2(\gamma^3\gamma^2\gamma^1\gamma^0)(\gamma^0\gamma^1\gamma^2\gamma^3) = -(\gamma^3)^2(\gamma^2)^2(\gamma^1)^2(\gamma^0)^2 = -(-1)(-1)(-1)(+1) = 1. \tag{11}$$

Comment: Observe the following. The properties we just derived tell us that, when we define $\gamma^4 = i\gamma^5$, the set $\gamma^M, M =, 0,1,2,3,4$ satisfy (1) in five dimensional space-time. In 4d however, γ^5 can serve a different purpose. Since $(\gamma^5)^2 = 1$, we have orthogonal projectors $P_{R/L} = \frac{1}{2}(1 \pm \gamma^5)$ onto the eigenspaces of γ^5 with eigenvalues ± 1. Spinors with eigenvalue $+1$ are called right-handed. Likewise, spinors with eigenvalue -1 are called left-handed. The massless Dirac equation

$$\gamma^\mu\partial_\mu\psi = 0 \tag{12}$$

decomposes into equations for left- and right-handed fermions. In 5d, $\gamma^4 = i\gamma^5$ enters itself in the Dirac equation, therefore there is no splitting in right-handed and left-handed components (this is true in any odd dimension). We cannot define a new γ^5 as we did in $4d$, since $\gamma^0 \cdots \gamma^4$ is proportional to the identity.

(iv) Let us define the Dirac conjugate to be a combination of hermitian conjugation and multiplication by γ^0:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0. \tag{13}$$

Verify that in our explicit representation of γ matrices (2) we have

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu \tag{14}$$

and in particular that γ^0 is hermitian. What does it imply about $(\gamma^5)^\dagger$? Why is it impossible to find a representation of γ^μ where γ^j would be hermitian?

Solution: Clearly, $\gamma^0\gamma^0\gamma^0 = \gamma^0$ and γ^0 is hermitian. Moreover,

$$\gamma^0\gamma^i\gamma^0 = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i^\dagger \\ \sigma_i^\dagger & 0 \end{pmatrix} = \gamma^i, \tag{15}$$

where we used that the Pauli matrices are hermitian. Since $(\gamma^0)^2 = 1$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$, it follows that the γ^i are anti-hermitian. For γ^5, this means that

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5, \tag{16}$$

so γ^5 is hermitian.

Assume now that we found a representation of the γ^μ such that all of them are hermitian. By a similar calculation as above, it follows that γ^5 is anti-hermitian in this case. This means that it has purely imaginary eigenvalues. On the other hand, we showed above that $\gamma^5 = 1$, independent of the representation of the γ^μ. But this contradicts that γ^5 has only imaginary eigenvalues.

2 Dirac equation

(i) The Dirac fermion is described by a 4-component vector of functions $\psi(x)$ on which the γ-matrices act by usual matrix multiplication. Show that the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{17}$$

implies that each component of ψ satisfies the Klein-Gordon equation. Hint: act on the Dirac equation with $(-i\gamma^\nu\partial_\nu - m)$.

Solution: Recall that the Klein-Gordon equation is $(\Box + m^2)\phi = 0$, where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the spacetime laplacian. As given in the hint, we act on the Dirac equation by $(-i\gamma^\nu\partial_\nu - m)$. This gives us

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\phi x = (\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi(x) = (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi(x) = (\Box + m^2)\psi(x). \tag{18}$$

Observe that $(\Box + m^2)$ acts diagonally on the four-component spinor $\psi(x)$. Therefore, if ψ satisfies the Dirac equation, each component of ψ satisfies Klein-Gordon.

(ii) Derive the Dirac equation for conjugate spinor $\bar\psi(x)$.

Solution: Recall that $\bar\psi(x) = \psi^\dagger\gamma^0$. To obtain an equation for $\bar\psi$, we conjugate the Dirac equation,

$$-i\partial_\mu\psi^\dagger(x)(\gamma^\dagger)^\mu - \psi^\dagger(x)m = 0. \tag{19}$$

We act on it with γ^0 from the right. This gives

$$0 = -i\partial_\mu\psi^\dagger(x)(\gamma^\dagger)^\mu\gamma^0 - m\psi^\dagger(x)\gamma^0 = -i\partial_\mu\psi^\dagger(x)\gamma^0\gamma^\mu - \psi^\dagger(x)\gamma^0 m = -i\partial_\mu\bar\psi(x)\gamma^\mu - \bar\psi(x)m, \tag{20}$$

where we used $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\dagger)^\mu$ from problem 1 (iv). The conjugate Dirac equation is sometimes written as

$$\bar\psi(x)(i\overleftarrow{\partial\!\!\!/} + m) = 0. \tag{21}$$

(iii) Derive the Dirac equation using the principle of the minimal (extremal) action from Lagrangian

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi. \tag{22}$$

Solution: The action is defined to be

$$S = \int \mathrm{d}^4 x\, \mathcal{L} = \int \mathrm{d}^4 x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi. \tag{23}$$

The Dirac fermion is a spinor with four complex components. To find the critical points of the action, we should therefore vary with respect to 8 variables. The easiest way to do this is to treat $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ as complex variables. We then vary with respect to these variables and their complex conjugates. The advantage is that when we vary with respect to ψ, the variation of $\psi^\dagger$ is zero, and vice versa. We first vary S with respect to $\psi^\dagger$. We find

$$\bar{\delta} S = \int \mathrm{d}^4\, x\, \delta\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{24}$$

Vanishing of $\bar{\delta} S$ gives the Dirac equation. Varying with respect to ψ gives

$$\delta S = \int \mathrm{d}^4\, x\, \bar{\psi}(i\gamma^\mu \partial_\mu - m)\delta\psi = \int \mathrm{d}^4\, x\, (-i\partial_\mu \bar{\psi}\gamma^\mu - \bar{\psi} m)\delta\psi. \tag{25}$$

This vanishes if and only if the conjugate Dirac equation is satisfied. We saw in the last problem that the Dirac equation implies the conjugate Dirac equation. Therefore, the critical points of S are the solutions to the Dirac equation.

3 Noether currents for Dirac equation

(i) (Noether's theorem) Assume that we have a Lagrangian $\mathcal{L}(\phi_j, \partial\phi_j)$ depending on fields and their first derivatives that is invariant under an infinitesimal transformation of the fields (up to a total derivative)

$$\mathcal{L}(\phi_j + \epsilon\delta\phi_j, \partial(\phi_j + \epsilon\delta\phi_j)) = \mathcal{L}(\phi_j, \partial\phi_j) + \epsilon\partial_\mu K^\mu(\phi_j, \partial\phi_j) + \mathcal{O}(\epsilon^2). \tag{26}$$

Show using the Euler-Lagrange equations that the current

$$J^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j - K^\mu \tag{27}$$

is conserved if the equations of motion are satisfied.

Solution: We differentiate (26) with respect to $\frac{\mathrm{d}}{\mathrm{d}\,\epsilon}|_{\epsilon=0}$. We find

$$\frac{\partial\mathcal{L}}{\partial\phi_j}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\partial_\mu\delta\phi_j = \partial_\mu K^\mu(\phi_j, \partial\phi_j). \tag{28}$$

We can use the equations of motion $\frac{\partial\mathcal{L}}{\partial\phi_j} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}$ on the left hand side,

$$\frac{\partial\mathcal{L}}{\partial\phi_j}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\partial_\mu\delta\phi_j = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\partial_\mu\delta\phi_j = \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j). \tag{29}$$

Equating this to the right hand side of (28) reads

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j) = \partial_\mu K^\mu. \tag{30}$$

Therefore,

$$\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j - K^\mu) = 0, \tag{31}$$

i.e. the current $J^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta\phi_j - K^\mu$ is conserved.

(ii) Notice that the Dirac action is invariant under global $U(1)$ transformations acting as

$$\psi(x) \mapsto e^{-iq\alpha}\psi(x). \tag{32}$$

where q is the charge of field ψ. Show that the corresponding Noether current is in this case

$$J^\mu = q\bar{\psi}\gamma^\mu\psi. \tag{33}$$

Verify that it is conserved if the Dirac equation of motion is satisfied.

Solution: The Dirac action

$$\int \mathrm{d}^4\, x\, \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{34}$$

is obviously invariant under phase shifts $\psi(x) \mapsto e^{-iq\alpha}\psi(x)$ since it only involves products of ψ with its complex conjugate. Since this does not involve total derivatives, $K^\mu = 0$ in the notation of (26). Infinitesimally, the transformation is

$$\delta\psi = \left.\frac{\mathrm{d}}{\mathrm{d}\,\alpha}\right|_{\alpha=0}(e^{-iq\alpha}\psi(x)) = -iq\psi(x), \quad \delta\bar{\psi} = \left.\frac{\mathrm{d}}{\mathrm{d}\,\alpha}\right|_{\alpha=0}(e^{iq\alpha}\bar{\psi}(x)) = iq\bar{\psi}(x). \tag{35}$$

Let us write ψ^a for the components of ψ. We obtain

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)}\delta\psi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu(\psi^*)^a)}\delta(\psi^*)^a = i\bar{\psi}^a\gamma^\mu_{ab}(-iq\psi^b(x)) + 0 = q\bar{\psi}\gamma^\mu\psi. \tag{36}$$

We used that $\mathcal{L}$ does not depend on the derivatives of the conjugate field when written in the form (34).

We write the Dirac equations as

$$\gamma^\mu\partial_\mu\psi = -im\psi \quad \text{and} \quad \partial_\mu\bar{\psi}\gamma^\mu = im\bar{\psi}. \tag{37}$$

Then,

$$\partial_\mu J^\mu = q((\partial_\mu\bar{\psi}\gamma^\mu)\psi + \bar{\psi}(\gamma^\mu\partial_\mu\psi)) = imq\bar{\psi}\psi - imq\bar{\psi}\psi = 0. \tag{38}$$

4 Local symmetries and QED

(i) Verify that the Dirac action is not invariant under local gauge transformations

$$\psi(x) \mapsto e^{-iq\alpha(x)}\psi(x) \tag{39}$$

but becomes invariant if we replace the derivative $\partial_\mu\psi$ by a covariant derivative

$$D_\mu\psi = (\partial_\mu + ieqA_\mu)\psi \tag{40}$$

where $A_\mu(x)$ is the gauge field and if we simultaneously transform the gauge field as

$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu\alpha. \tag{41}$$

Solution: The Dirac equation transforms under $\psi(x) \mapsto e^{-iq\alpha(x)}\psi(x)$ as

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \mapsto (i\gamma^\mu\partial_\mu - m)e^{-iq\alpha(x)}\psi(x) = e^{-iq\alpha(x)}(i\gamma^\mu\partial_\mu + q\gamma^\mu\partial_\mu\alpha(x) - m)\psi(x) = 0. \tag{42}$$

We see that if $\psi(x)$ satisfies the Dirac equation, $e^{-iq\alpha(x)}\psi(x)$ will no longer satisfy it unless α is a constant. Equivalently, we could argue on the level of the action. The Lagrangian of the free fermion is not invariant under $\psi(x) \mapsto e^{-ie\alpha(x)}\psi(x)$. We postpone the question involving the gauge field to the next question.

(ii) How does $D_\mu\psi$ transform under the gauge transformations?

Solution:

$$D_\mu\psi = (\partial_\mu + ieqA_\mu)\psi \mapsto (\partial_\mu + ieqA_\mu + \frac{iqe}{e}\partial_\mu\alpha)e^{-iq\alpha(x)}\psi(x) \tag{43}$$

$$= e^{-iq\alpha(x)}(\partial_\mu + ieqA_\mu + iq\partial_\mu\alpha - iq\partial_\mu\alpha)\psi(x) = e^{-iq\alpha(x)}D_\mu\psi. \tag{44}$$

We say that $D_\mu\psi$ transforms covariantly[1] with respect to the gauge transformation. When we use the covariant derivative in the Dirac equation, we find that it is indeed invariant, since

$$(i\gamma^\mu D_\mu - m)\psi(x) = 0 \mapsto e^{-iq\alpha(x)}(i\gamma^\mu D_\mu - m)\psi(x) = 0. \tag{45}$$

(iii) Show that the replacement of an ordinary derivative by the covariant one is equivalent to additional coupling of the form $-eJ^\mu A_\mu$ in the Lagrangian where J^μ is the Noether current that we found previously.

[1] This means that $D_\mu\psi$ transforms like ψ.

Solution: The Dirac Lagrangian with covariant derivative is

$$\mathcal{L}_{cov} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - eq\gamma^\mu A_\mu - m)\psi(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) - eq\bar{\psi}(x)\gamma^\mu A_\mu\psi(x) \tag{46}$$

$$= \mathcal{L}_{cov}|_{A^\mu=0} - eA_\mu q\bar{\psi}(x)\gamma^\mu\psi(x). \tag{47}$$

In problem 3 (ii), we found the Noether current $J^\mu = q\bar{\psi}(x)\gamma^\mu\psi(x)$. Therefore, we have indeed

$$\mathcal{L}_{cov} - \mathcal{L}_{cov}|_{A^\mu=0} = -eJ^\mu A_\mu. \tag{48}$$

(iv) Let us define the curvature (electromagnetic tensor) $F_{\mu\nu}$ such that

$$[D_\mu, D_\nu]\,\psi(x) = iqeF_{\mu\nu}(x)\psi(x). \tag{49}$$

Express $F_{\mu\nu}(x)$ in terms of A_μ and find how it transforms under the gauge transformations.

Solution:

$$[D_\mu, D_\nu]\psi = [\partial_\mu + ieqA_\mu, \partial_\nu + ieqA_\nu]\psi = ieq([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi = ieq(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi. \tag{50}$$

When we compare this to (49), we see that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under the gauge transformation $A\mu \mapsto \frac{1}{e}\partial\mu\alpha$, the curvature transforms as

$$F_{\mu\nu} \mapsto \partial_\mu A_\nu + \frac{1}{e}\partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \frac{1}{e}\partial_\nu\partial_\mu\alpha = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \tag{51}$$

so it is in fact gauge invariant.

(v) The full Lagrangian of QED with Dirac matter is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi. \tag{52}$$

Find the equations of motion.

Solution:. We need to vary the action both with respect to A_μ and ψ. When we vary with respect to A_μ, we find

$$\delta_A S = \int \mathrm{d}^4\,x - \frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} - eJ^\mu\delta A_\mu = \int \mathrm{d}^4\,x - F^{\mu\nu}\partial_\mu\delta A_\nu - eJ^\mu\delta A_\mu = \int \mathrm{d}^4\,x(\partial_\mu F^{\mu\nu} - eJ^\nu)\delta A_\nu. \tag{53}$$

We find $\partial_\mu F^{\mu\nu} = J^\nu$ as one equation of motion. These are in fact the inhomogeneous Maxwell equations. On the other hand, variation with respect to ψ gives the covariant Dirac equation $(i\gamma^\mu D_\mu - m)\psi$ (the computation is the same as in problem 2 (iii), where we covered the non-covariant case).