Arnold Sommerfeld Center	Ludwig–Maximilians–Universität München

Prof. Ivo Sachs

Summer term 2019

Quantum Field Theory – Exam
22. Juni 2019

Please fill in:

Name: ______________________________

Matriculation Number: ______________________________

Number of Sheets: ______________________________

Please read carefully:

- If you do *not* want your score with your matriculation number on the webpage, please mark with a cross: ☐
- Please write your name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each indvidual problem.
- You have 150 minutes to answer the questions.
- *No* resources are allowed.
- Use a blue or black permanent pen.

Do not write below this line.

Comments:

Problem 1	/ 29 pt
Problem 2	/ 18 pt
Problem 3	/ 19 pt
Total	/ 66 pt

1 Short questions

(i) (2pt) Write down the free massive Dirac equation for a massive fermion ψ. Show that the solution of the Dirac equation satisfies the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi = 0. \tag{1}$$

(ii) (3pt) What is the dimension of the adjoint representation of the Lie algebra $\mathfrak{su}(2)$? Show that the matrices $(T_a)^b{}_c \equiv iC_{ac}{}^b$ satisfy the commutation relations of the Lie algebra, $[T_a, T_b] = iC_{ab}{}^c T_c$.

(iii) (4pt) Consider the non-abelian gauge theory with matter field ϕ and the covariant derivative

$$D_\mu \phi = \partial_\mu \phi + igA_\mu^a T_a \phi. \tag{2}$$

How is the curvature (field strength) $G^a_{\mu\nu}$ associated to gauge field A^a_μ defined? How do the gauge field and the curvature tensor transform under infinitesimal gauge transformations?

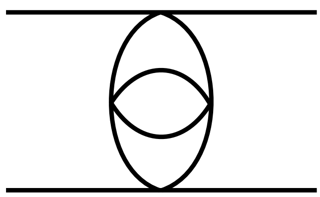
(iv) (2pt) Consider a free real scalar field ϕ. Draw the Feynman diagrams for the four-point correlation function. Which additional Feynman diagram(s) contribute to this correlation function at tree level if we add a ϕ^4 interaction term to the Lagrangian?

(v) (6pt) Show that, due to gauge invariance, the kinetic term of a $U(1)$ gauge theory is not invertible. Add a gauge-fixing term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to the Lagrangian and derive the propagator [If you have problems for general ξ, try at least $\xi = 1$].

(vi) (4pt) Write down the minimal BV action for free Maxwell theory. What is the degree (ghost number) and parity (statistics) of the fields? Show that the classical master equation is satisfied. What is the BRST transformation of the gauge field A_μ?

(vii) (2pt) Explain why ghosts decouple if you consider Maxwell theory in Lorenz gauge.

(viii) (1pt) Determine the symmetry factor of the following Feynman diagram in ϕ^4 theory:



(ix) (4pt) Consider the following formal expression for the propagator of a scalar field

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2}. \tag{3}$$

As it is written, the integral is not well-defined. Make a sketch of the pole structure and the integration contour in the complex p^0 plane. Explain the choice of integration contour (i.e. the role of the Feynman $i\epsilon$ prescription) leading to Feynman propagator and its relation to Wick rotation. How should we shift the poles to find instead the retarded Green function (i.e. Green function vanishing for $x^0 < 0$)?

(x) (1pt) What is the generating functional for connected correlation functions?

2 Scalar QED

Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{\lambda}{6} |\phi|^4, \tag{4}$$

where ϕ is a complex scalar field.

(i) (2pt) Find the equations of motion for ϕ.

(ii) (1pt) The theory has one continuous internal $U(1)$ symmetry. Find it.

(iii) (2pt) Derive the Noether current associated to the symmetry you found.

(iv) (2pt) Verify that the current you found is conserved if the equations of motion are satisfied.

(v) (4pt) Gauge the $U(1)$ symmetry of the theory – use the minimal coupling to couple the scalar field to a electromagnetic field. Write down the resulting Lagrangian (including the kinetic term for the gauge field). Write down the local symmetry transformation under which the Lagrangian is invariant.

(vi) (3pt) Find the classical equations motion for all fields in this theory (including the gauge field).

(vii) (2pt) Compare the Noether current obtained from the original Lagrangian (before gauging it) to the current appearing on the RHS of Maxwell's equations that you found now. Which of them is gauge invariant?

(viii) (2pt) Sketch the Feynman diagrams describing interactions of the theory. Which interaction term is new and isn't there in QED with fermionic matter?

3 Finite Dimensional BRST-BV

Consider the following integral

$$\langle f \rangle = \int_{\mathbb{R}^2} \mathrm{d}\,x\,\mathrm{d}\,y e^{-S_0(x,y)} f(x-y), \tag{5}$$

with action $S_0(x,y) = \frac{1}{2}(x-y)^2$.

(i) (2pt) Identify the "gauge" symmetry of this action – write down explicitly the transformation(s) under which the action is invariant.

(ii) (5pt) Use the Faddeev-Popov trick to gauge fix this action, with $F(x,y) = x + y + G(x-y)$ as a gauge fixing condition, where $G(z)$, $z \in \mathbb{R}$, is some arbitrary regular function. You should find

$$S_0 + S_{gh} = \frac{1}{2}(x-y)^2 + C\eta^*\eta, \tag{6}$$

with some (specific) constant C, and (η^*, η) a pair of fermionic variables. The complete gauge fixed integral should be

$$\int \mathrm{d}\,x\,\mathrm{d}\,y f(x-y) e^{-S_0 - S_{gh}} \delta(F(x,y)) \tag{7}$$

(iii) (2pt) Average the Gauge fixing condition $F(x,y) = c$ over c with the weight function $e^{-\frac{1}{2}c^2}$. You should find

$$S_{tot} = \frac{1}{2}(x-y)^2 + C\eta^*\eta + \frac{1}{2}F^2. \tag{8}$$

(iv) (4pt) Write down the concrete expressions of the BRST transformations $\delta_B \phi = \zeta \mathfrak{s}(\phi)$ for all the four variables, $\phi = (x, y, \eta, \eta^*)$. Show that S_{tot} is invariant under them.

(v) (3pt) The BRST transformations do not yet square to zero. Use

$$\int e^{-b^2 + ibF} db \sim e^{-F^2} \tag{9}$$

to integrate in a Lagrange multiplier b and write down the new BRST transformations. Show that $\delta_B^2 = 0$.

(vi) (3pt) Go back to the original S_0. Find its (minimal) BV extended action and show that it satisfies the classical master equation (without using the equations of motion).