3) BRST Quantisation

Ward-Takahashi-Slavnov-Taylor identities (Eq. Itzykson - Zuber)

If j_μ is a conserved current due to some continuous symmetry α, then

$$0 = \int \partial_{y^\mu} \langle 0 | T\{ j^\mu(y)\, O(x_1) \cdots O(x_n) \} | 0 \rangle \, d^4y$$

$$\sum \langle 0 | O(x_1) \cdots \delta_\alpha O(x_i) \cdots O(x_n) | 0 \rangle$$

where $\delta_\alpha O$ is the variation of O under a global symmetry transformation with parameter α.

Ward-Takahashi id

Pf.: Let Q be the Noether charge associated to the symmetry α of a Lagrangian system. If L is Lorentz covariant we then

$$Q = \int d^3x \; j_0(\underline{x}, t)$$

for some conserved current $j_\mu(\underline{x}, t)$

In the quantum theory the symmetry α translates into the condition

$$Q|0\rangle = 0$$

Thus $0 = \langle 0 | [Q, V_1(\underline{x}_1, t_1) \cdots V_n(\underline{x}_n, t_n)] | 0 \rangle$

$$= \int \langle 0 | [j_0(\underline{y}, t), V_1(\underline{x}_1, t_1) \cdots V_n(\underline{x}_n, t_n)] | 0 \rangle \, d^3y$$

$$= \int d^3y \, dt \, \partial_{y^\mu} \langle 0 | T\{ j^\mu(\underline{y}, t) V_1(\underline{x}_1, t_1) \cdots V_n(\underline{x}_n, t_n) \} | 0 \rangle$$

$$\partial_\mu j^\mu = 0 \, ; \quad \partial_t \, \theta(t - t_i) = \delta(t - t_i)$$

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Rem:

We can also consider a local transformation by letting α depend on x. Then we have instead the weaker condition:

$$- \int \partial_\mu \alpha \, \langle 0 | T\{ j^\mu(y) \, G(x_1) \cdots G(x_n) \} | 0 \rangle$$

$$+ \sum_i \langle 0 | G(x_1) \cdots S_\alpha G(x_i) \cdots G(x_n) | 0 \rangle = 0.$$

What is the analog of this in the FP quantisation of some non-abelian gauge theory? Since we already fixed the gauge one might think that there is no trace of such WI.
On the other hand we have argued that the FP-procedure is independent of the gauge choice.

For a generic, not-necessarily gauge-invariant insertion, $f(A_\mu)$ we then define

$$\langle f(A_\mu)\rangle := \frac{1}{Z}\int D[A_\mu]\,\Delta[A_\mu]\,f(A_\mu)\,e^{\frac{i}{\hbar}S[A_\mu]+\frac{i}{2\gamma}\int_{\mathbb{R}^4}F^a(A_\mu)F^a(A_\mu)}$$

∴ note that this definition is <u>not</u> unique. For instance replacing $F^a(A_\mu)$ by $F^a(A_\mu^u)$ would be another definition
Under a infinitesimal gauge transformation

$$A_\mu \to A_\mu' = A_\mu + \frac{1}{g}D_\mu^{adj}\theta$$

However, since we integrate over the whole configuration space $\{A_\mu\} = \{A_\mu'\}$ we have

$$\langle f(A_\mu') \rangle = \frac{1}{Z} \int D[A_\mu'] \, \Delta[A_\mu'] f(A_\mu') \, e^{\frac{i}{\hbar}\left(S[A_\mu'] + S_{gf}(A_\mu')\right)}$$

$$= \langle f(A_\mu) \rangle$$

On the other hand, since

$$D[A_\mu'] \, \Delta[A_\mu'] \, e^{\frac{i}{\hbar} S[A_\mu']} = D[A_\mu] \, \Delta[A_\mu] \, e^{\frac{i}{\hbar} S[A_\mu]}$$

we conclude that, for an infinitesimal transformation,

$$0 = \int D[A_\mu] \, \Delta[A_\mu] \left(\underbrace{\delta f(A_\mu)}_{= f(A_\mu^u) - f(A_\mu)} + f(A_\mu) \frac{i}{\gamma} \int F^a \underbrace{\delta F^a}_{F(A_\mu^u) - F(A_\mu)} \right) \cdot e^{\frac{i}{\hbar} S[A] + \frac{i}{2\gamma} \int F^a F^a} \qquad (\square)$$

This is a <u>Slavnov-Taylor identity</u> which reproduces the Ward-Takahashi Identity above. Indeed for a global transformation $\theta \neq \theta(x)$ $\delta(F^a F_a) = 0$ and thus $\langle \delta f(A_\mu) \rangle = 0$ reproduces the WI above.

The analogy with the previously reviewed Ward-Takahashi identity becomes apparent if we consider a multilocal insertion $f(A_\mu)(x_1, \cdots, x_n)$ and recall that Euclidean correlation functions translate into time ordered correlation functions in Minkowski signature.

An alternative representation of this identity is obtained in terms of the ghost fields η^{+a}, η^a: let us write

$$S^{tot} = \int d^4x \left(h(A_\mu) + \frac{\lambda}{2} F^a F^a + \bar{H}_a M^a{}_b H^b \right)$$

with $\lambda = \frac{\hbar}{\gamma}$. Then one can show that S^{tot} is invariant under the local transformation .

$$(*) \quad \begin{cases} \delta A^\mu_a(x) = \delta\xi \, D^\mu{}_{ab} H^b =: \delta\xi \, s(A^\mu_a)(x) \\ \delta \bar{H}_a(x) = -\delta\xi \, \lambda F_a(A, x) =: \delta\xi \, s(\bar{H}_a)(x) \\ \delta H^a(x) = \delta\xi \, \frac{1}{2} C^a{}_{bc} H^b(x) H^c(x) =: \delta\xi \, s(H^a)(x) \end{cases}$$

where s is an odd vector field on the graded space of fields and $\delta\xi$ is a constant, odd (anti-commuting) parameter. We leave the proof of this as an exercise. The invariance under $(*)$ is the BRST invariance of the DeWitt-Faddeev-Popov path integral.