Feynman rules for QCD in Lorentz gauge

dep. on gauge!

$$S^{(2)}(A_\mu,\psi,\bar\psi,H,\bar H) = \int \bar\psi(i\not\partial - m)\psi$$

$$-\frac{1}{4}\int \underbrace{B_{ab}}_{=\delta_{ab}} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu b} - \partial^\nu A^{\mu b})$$

$$+\frac{\hbar}{2\gamma}\int \partial_\mu A^{a\mu}\partial^\nu A^b_\nu B_{ab}$$

$$+\frac{1}{2} A^a_\mu\left(\Box - \left(1+\frac{\hbar}{\gamma}\right)\partial^\mu\partial_\nu\right)\delta_{ab}A^{\nu b}$$

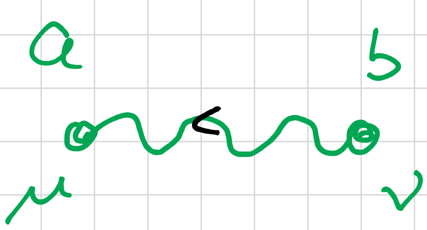
$$+\frac{\hbar}{g}\int \bar H_a(x)\,\Box\,\delta^a{}_b\, H^b(x)$$

this factor can be modified at will by a rescaling of H, and $\bar H$

$\equiv 1$

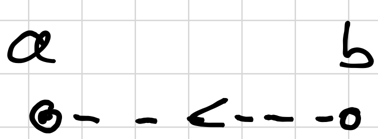
Propagators:

$$S_F(p,m) = -i\hbar\,\frac{\not p + m}{p^2 - m^2 + i\varepsilon}$$



$$D^{ab}_{\mu\nu}(p) = \frac{i\hbar\,\delta^{ab}}{p^2+i\varepsilon}\left(\eta_{\mu\nu} - \left(1+\frac{\gamma}{\hbar}\right)\frac{p_\mu p_\nu}{p^2+i\varepsilon}\right)$$

Exercise (sheet 4)



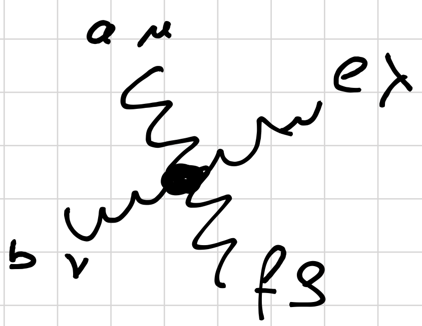
$$G^{ab}(p) = \frac{-i\,\delta^{ab}}{p^2+i\varepsilon}$$

Interaction vertices:

$$h_I = -\frac{g}{2}\,\delta_{ae}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) C_{cd}{}^{e} A^{\mu c} A^{\nu d}$$

$$+\frac{g^2}{4}\, C_{ab}{}^{d}\, C_{ef}{}^{h}\, \delta_{dh}\, A_\mu^a A_\nu^b A^{\mu e} A^{\nu f}$$

$$-\, g\,\bar{\psi}\gamma^\mu T_a \psi\, A_\mu^a + g\left(\partial_\mu \bar{H}_a\right) C_{ed}{}^{a} H^d A^{\mu e}$$

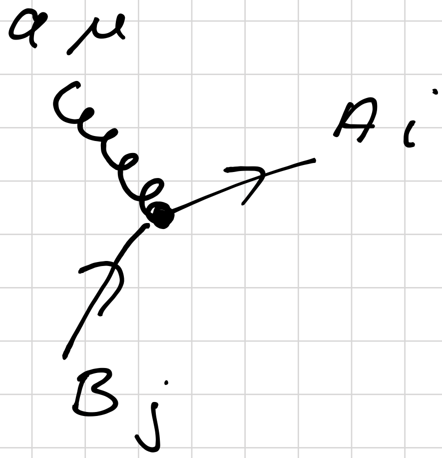
$\rightarrow$ 

$$= \frac{i}{\hbar} g^2 (2\pi)^4 \delta^4\left(\sum_{i=1}^{4} p_i\right) \cdot$$

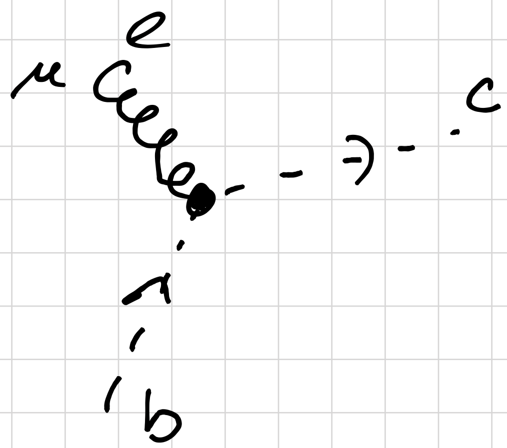
$$\Big[C_{ab}{}^{z} C_{efz}\left(\eta^{\mu\lambda}\eta^{\nu\varsigma} - \eta^{\nu\lambda}\eta^{\mu\varsigma}\right) + C_{ae}{}^{z} C_{bfz}\left(\eta^{\lambda\varsigma}\eta^{\mu\nu} - \eta^{\lambda\nu}\eta^{\mu\varsigma}\right)$$

$$+ C_{af}{}^{z} C_{bez}\left(\eta^{\lambda\varsigma}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\nu\varsigma}\right) \Big]$$

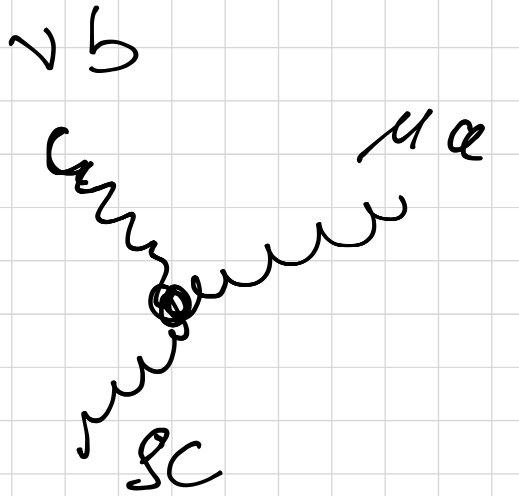
$$\boxed{\begin{aligned}
& C_{a\nu}{}^{z} A_\beta^\nu A_\mu^o A_\beta^p C_{opz} + A_\alpha^u C_{ua}{}^{z} A_\alpha^o A_\mu^p C_{opz} \\
& + C_{uv}{}^{z} A_\mu^u A_\beta^\nu C_{ap}{}^{z} A_\beta^p + C_{uv}{}^{z} A_\alpha^u A_\mu^\nu A_\alpha^o C_{oa}{}^{z} \\
& C_{ab}{}^{z} A_\mu^o A_\nu^p C_{op}{}^{z} + C_{a\nu}{}^{z} A_\beta^\nu \eta^{\mu\nu} C_{b\beta z} A_\beta^p + C_{a\nu}{}^{z} A_\nu^\nu A_\mu^o C_{obz} \\
& C_{ab}{}^{z} \eta^{\lambda\mu}\eta^{\nu\varsigma} C_{efz} + \eta^{\lambda\nu}\eta^{\varsigma\mu} C_{ab}{}^{z} C_{fez} + \eta^{\lambda\varsigma}\eta^{\mu\nu}\left(C_{ae}{}^{z} C_{bfz} + C_{af}{}^{z} C_{bez}\right) \\
& + \eta^{\lambda\nu} C_{ae}{}^{z} \eta^{\varsigma\mu} C_{fbz} + \eta^{\mu\lambda}\eta^{\nu\varsigma} C_{afz} C_{ebz}
\end{aligned}}$$



$$-\frac{gi}{\hbar}(\gamma_\mu)_{AB}(T_a)_{ij}(2\pi)^4\delta^4\Big(\sum_i^3 p_i\Big)$$



$$-\frac{g}{\hbar}p_\mu^{(b)}\, C_{ebc}\,(2\pi)^4\,\delta^4\Big(\sum_{i=1}^3 p_i\Big)$$



$$g\, C_{abc}\,(2\pi)^4\,\delta\Big(\sum_{i=1}^3 p_i\Big)\Big(g_{\mu\nu}(p^{(a)}-p^{(b)})_\nu + g_{\nu\rho}(p^{(b)}-p^{(c)})_\nu + g_{\rho\mu}(p^{(c)}-p^{(a)})_\nu\Big)$$

Exercise (Sheet 4)

Litt: Pokorski, Itzykson, Zuber, ...

Unitarity of the S-matrix

Top: QED: (from my QM II notes)

In **Coulomb gauge**,

$$\nabla \cdot \vec{A}(\vec{q}, t) = 0 \tag{2.32}$$

with $A_0 \equiv 0$

and in the absence of sources, the **vector potential** solves the wave equation $\Box \vec{A}(\vec{q}, t) = 0$ and is related to the electro-magnetic field through

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \wedge \vec{A} \tag{2.33}$$

We take the cavity to be a cube of length L and assume **periodic boundary conditions** for the electromagnetic field, $\vec{A}(\vec{q} + \vec{n}L, t) = \vec{A}(\vec{q}, t)$ for $\vec{n} \in \mathbf{Z}^3$. The general solution to the wave equation is the given by (classical ED, eg. Jackson)

$$\vec{A}(\vec{q}, t) = \frac{1}{\sqrt{L^3}} \sum_{\vec{k} = \frac{2\pi}{L}\vec{n}} \sum_{\lambda=1}^{2} \vec{e}(\vec{k}, \lambda) \left(a(\vec{k}, \lambda) e^{-i(\omega(\vec{k})t - \vec{k}\cdot\vec{q})} + a^*(\vec{k}, \lambda) e^{i(\omega(\vec{k})t - \vec{k}\cdot\vec{q})} \right) \tag{2.34}$$

with $\omega(\vec{k}) = c|\vec{k}|$ and real polarisation vectors $\vec{e}(\vec{k}, \lambda)$, orthogonal to $\vec{k}$ and

$$\vec{e}(\vec{k}, \lambda) \cdot \vec{e}(\vec{k}, \lambda') = \delta_{\lambda\lambda'} \tag{2.35}$$

A straight forward calculation shows that (in Gaussian units) the classical energy of the electromagnetic field is then given by

$$h(\vec{E}, \vec{B}) = \frac{1}{8\pi} \int (\vec{E}^2 + \vec{B}^2) d^3q = \frac{1}{2\pi} \sum_{\vec{k}} \sum_{\lambda=1}^{2} |a(\vec{k}, \lambda)|^2 \tag{2.36}$$

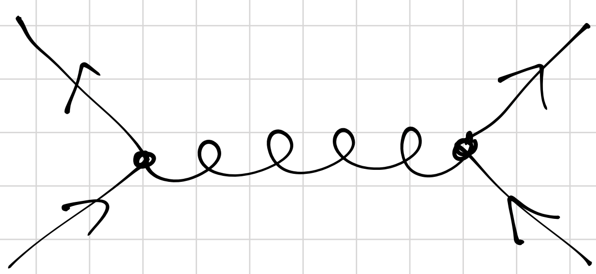
Starting from (2.34) and defining the real variables

From this we see that the em field propogates 2 polarisations. On the other hand, the propagator obtained from the Faddeev-Popov procedure propagates 4-polarisations. Indeed

$D^{ab}_{\mu\nu}(k) \quad =$ 

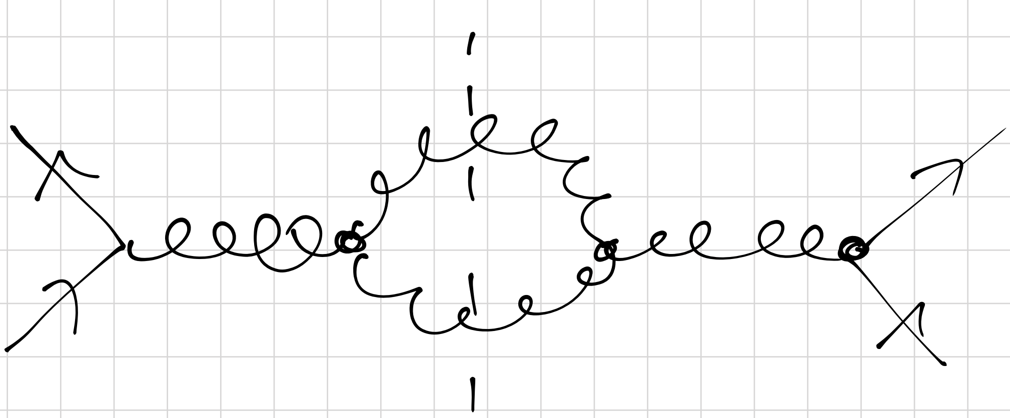
does not project on any subspace (except for the Landau gauge $\xi \to 0$)

How is this compatible with unitarity in e.g.



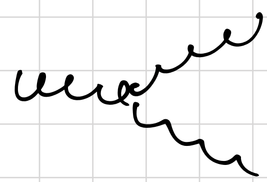
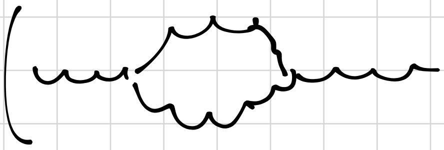
This diagram turns out to be unproblematic since current conservation ensures that only transverse photons propagate.

In QED this is the only way photons can couple so there is no issue. (This is also why QED can be quantised without ghost) In QCD or non-abelian gauge theories however, another diagram contributes as well.



If we cut this diagram

this appears to be in contradiction with the optical theorem:

$\Bigg|$  $\Bigg|^2$ = Im ()

<u>Proof:</u> unitarity $\Rightarrow$ $1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$

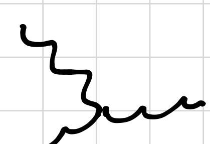
$$= 1 + i(T - T^\dagger) + T^\dagger T$$

writing $\langle f|T|i\rangle = (2\pi)^4 \delta(p_f - p_i)\, F_{fi}$ this gives

$$(F_{fi} - F_{if}^*) = i \sum_n (2\pi)^4 \delta(p_n - p_i)\, F^*_{nf} F_{ni}$$

$\uparrow$ intermediate states

For $|f\rangle = |i\rangle$ (forward scattering) the l.h.s $= \mathrm{Im}(T_{ii})$

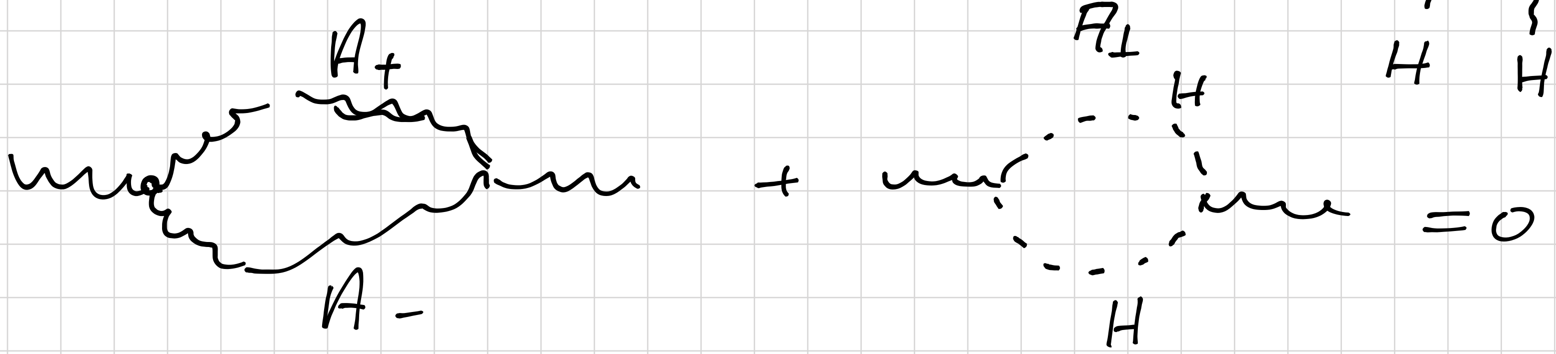
The r.h.s. vanishes  $= 0$ by momentum conservation but the l.h.s does not. as can be seen using e.g. $\mathrm{Im}(D^{ab}_{\mu\nu}) = \mathrm{Im}\left(\delta_{ab} \frac{g_{\mu\nu}}{k^2 + i\varepsilon}\right)$

$\sim -2\pi i\, g_{\mu\nu} \delta_{ab} \delta(k^2)\, \Theta(k^0)$. Then l.h.s.

$\sim \log\left(\frac{P^2}{\mu^2}\right) \neq 0$

However, it turns out that the ghosts contribute exactly the opposite contribution thus cancelling the unphysical states. (factors of g^2)

Qualitatively, $A_\mu = \big(\underbrace{\underline{A} : \underline{\nabla}\cdot\underline{A}=0}_{A_\perp}\big), \underbrace{A_+}_{H}, \underbrace{A_-}_{H^*}$



Historically, Feynman introduced the ghosts $H, \bar{H}$ for that purpose.

It is a nice exercise to check this explicitely using the above vertices and propagators.