Rem: $(i\not{\partial} - m)(i\not{\partial} + m) = -\not{\partial}^2 - m^2$

$= \Delta - m^2$

$$\{\gamma_E^\mu, \gamma_E^\nu\} = \ominus 2\delta^{\mu\nu}$$

$(+,-\,-\,-)$

cf $\{\gamma_M^\mu, \gamma_M^\nu\} = 2\cancel{g}\,\eta^{\mu\nu}$

Thus: $K(\underline{x},\underline{y}) = \langle \underline{x}| \dfrac{i\not{\partial} + m}{\Delta - m^2} |\underline{y}\rangle$

Wick rotation $t = e^{\frac{\pi i}{2}} \tau$; $\gamma^0_E \to e^{\frac{\pi i}{2}} \gamma^0_E = \gamma^0_M$

with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$$S_E[\psi] = \int \psi^* (i\cancel{\partial} - m)\psi$$

$$\to \quad S[\psi] = \int \bar{\psi} (i\cancel{\partial} - m)\psi \quad ; \quad \bar{\psi} = \psi^\dagger \gamma^0$$

$$K(x,y) \to S_F(x,y) \equiv -i\hbar \, K_M(x,y)$$

$$= -i\hbar \langle x | \frac{i\cancel{\partial} + m}{-\Box - m^2 + i\varepsilon} | y \rangle$$

$$= \langle 0 | T\{\psi(\underline{x}, x_0) \bar{\psi}(\underline{y}, y_0)\} | 0 \rangle$$

$$\equiv \Theta(x_0 - y_0) \langle 0 | \psi(\underline{x}, x_0) \bar{\psi}(\underline{y}, y_0) | 0 \rangle$$

$$- \; \Theta(y_0 - x_0) \langle 0 | \bar{\psi}(\underline{y}, y_0) \psi(\underline{x}, x_0) | 0 \rangle$$

In momentum space:

$$S_F(p, m) = -i\hbar \frac{\cancel{p} + m}{p^2 - m^2 + i\varepsilon}$$

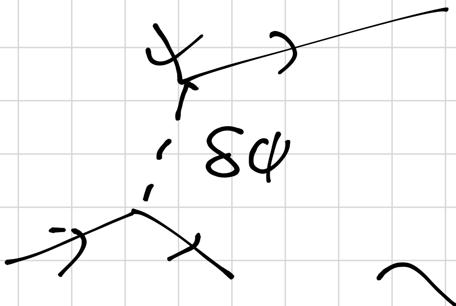
Interactions

① Yukawa couplings $\quad \mathcal{L}_{int} = g\bar{\Psi}\phi\Psi$

ϕ ← Higgs

$\phi = \phi_0 + \delta\phi$ ← fluct → interaction

ϕ_0 = const → mass for Ψ



↝ not discussed in this lecture

② gauge interactions : (canonical)

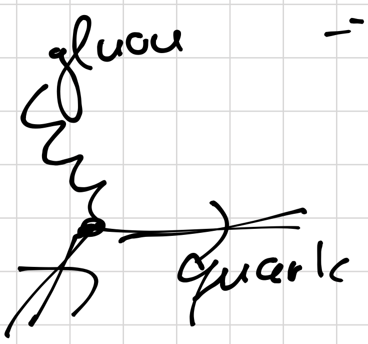
Rep: $\quad \bar{\Psi}(i\not{\partial} - m) \to \bar{\Psi}(i\not{D} - m)\Psi$

$\not{D} = \gamma^\mu(\partial_\mu + igA_\mu)$

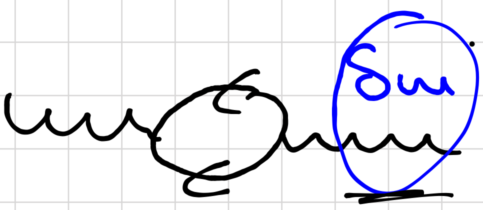
$\to \mathcal{L}_{int} = g\bar{\Psi}\gamma^\mu\Psi A_\mu \quad$ QED

T^a ← generator of gauge gr

$= g\underbrace{\bar{\Psi}\gamma^\mu T^a\Psi}_{=j^{\mu a}} A_\mu^a \quad$ QCD



Example: $\langle j_\mu(x)\, j_\nu(y) \rangle$: QED



$$= \langle :\bar\psi(x)\gamma_\mu \psi(x): \bar\psi(y)\gamma_\nu \psi(y): \rangle$$

$$= -\langle \psi_{B_2}(y)\, \bar\psi_{A_1}(x) (\gamma_\mu)_{A_1A_2}\, \psi_{A_2}(x)\, \bar\psi_{B_1}(y) (\gamma_\nu)_{B_1B_2} \rangle$$

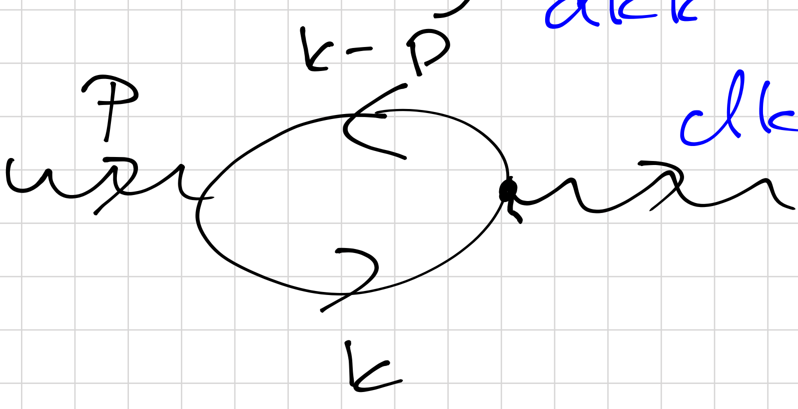
$$S_F(y,x)_{B_2A_1} (\gamma_\mu)_{A_1A_2}\; S_F(x,y)_{A_2B_1} (\gamma_\nu)_{B_1B_2}$$

$$= -\mathrm{tr}\left(S_F(y,x)\, \gamma_\mu\, S_F(x,y)\, \gamma_\nu \right)$$

in momentum space

$$= \frac{1}{2}\left(\frac{-ig}{\hbar}\right)^2 (-i\hbar)^2 \int d^4k \;\mathrm{tr}\left(\frac{\not k + m}{k^2-m^2+i\varepsilon}\, \gamma_\mu T^a \,\frac{(\not k - \not p + m)}{(k-p)^2-m^2+i\varepsilon}\, \gamma_\nu T^b \right)$$

$dk\, k^3$



$dk\, k^3\, \frac{1}{k^2}$

Rem: ① In a non-abelian g.t. $\gamma_\mu \to \gamma_\mu T^a$

② power counting $\sim \int dk\, k \sim \Lambda^2$ ← cutoff

here δm^2 (cancelling the quadr. div.)
is actually absent due to the conservation
$\partial_\mu j^\mu = 0$: I.e $\exists$ a regularisation
of $\int d^4k \, (\ldots)$ that preserves

$\partial_\mu j^\mu = 0 \iff$ gauge invariance

Rep: $(A_\mu + \partial_\mu \alpha) j^\mu \sim A_\mu j^\mu$

$\uparrow$
$\partial_\mu j^\mu = 0$

If such a regularisation does not
exist then the gauge symm. would
anomalous as it happens typically
for chiral fermions.

$\delta m^2 \neq 0$:

$$S[A_\mu] \to S[A_\mu] + \underbrace{\int d^4x \, \delta m^2 A_\mu A^\mu}_{\text{counter term}}$$

not invariant under $A_\mu \to A_\mu + \partial_\mu \alpha$

$\phi = \phi_0 + \delta\phi$ in Higg-

Quantisation of gauge fields

consider

$$\langle \underline{O(\psi, \bar\psi, A_\mu)} \rangle = \frac{1}{Z} \cdot$$

$$\cdot \int [D\psi, \bar\psi] [DA_\mu] \, O(\psi, \bar\psi, A_\mu) \, e^{\frac{i}{\hbar} S[A_\mu, \psi, \bar\psi]}$$

some gauge invariant op.

e.g. $O(\psi, \bar\psi) = B_{ab} :\underbrace{\bar\psi \gamma^\mu T^a \psi}_{j^{a\mu}}:(x) :\bar\psi \gamma^\mu T^b \psi(y):$

$$\delta j^{\mu a} = C^a{}_{bc} \, j^{\mu c} \, \theta^b$$

(QED)

$$O(\psi, \bar\psi, A_\mu) = \underline{\bar\psi(x)} \, \underbrace{e^{ie\int_x A_\mu ds^\mu}}_{\text{wilson line}} \, \underline{\psi(y)}$$

$$\bar\psi \, U^\dagger(x) \qquad U(y)\psi$$

Assume $[DA_\mu^U] = [DA_\mu]$ gauge inv. of the measure

$\Longleftrightarrow$ no anomalies

$\underset{\text{Rep}}{=} \frac{1}{ig} U \partial_\mu U^{-1} + U A_\mu U^{-1}$

Quantization $\quad \underbrace{S[\phi]}_{\text{symm.}} \,, \underbrace{[\bar{D}\phi]}_{\text{symm.}}$

then

$$\int \underbrace{[DA_\mu]}_{\text{inv}}\underbrace{[D\psi,\bar\psi]}_{\text{inv}}\ \underbrace{O(\psi,\bar\psi,A_\mu)}_{\text{inv.}}\ e^{\frac{i}{\hbar}\underbrace{S[\psi,\bar\psi,A_\mu]}_{\text{inv}}}$$

$$= \underbrace{\int [DA_\mu^u]}_{\int [Du] \int [D\underbrace{[A_\mu]}_{\text{equivalence classes}}]}\ \underbrace{[D\psi^u\bar\psi^u]\ O(\psi^u,\bar\psi^u,A_\mu^u)\cdot e^{\frac{i}{\hbar}S[\bar\psi^u,\bar\psi^u,A_\mu^u]}}_{\int [D\psi\bar\psi]\ O(\psi,\bar\psi,A_\mu]\ e^{\frac{i}{\hbar}S[\psi,\bar\psi,A_\mu]}\ \text{only depends on } [A_\mu]\ \text{equiv. cl.}}$$

orbits of gauge tf.

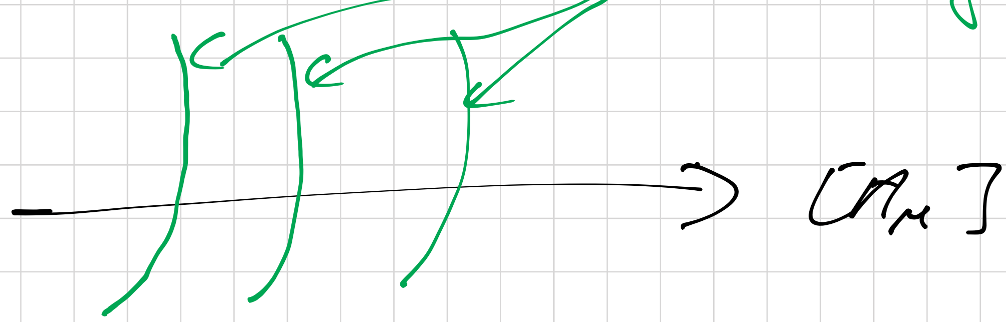


Illustration: (particle)

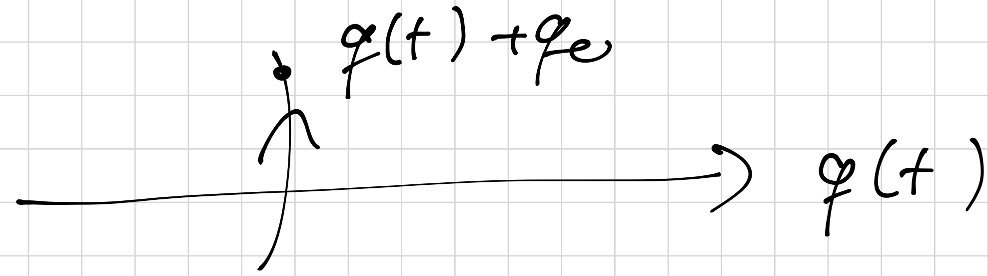
$$S[\dot q] = \int_0^T \frac{1}{2} m\, \dot q(t)^2\, dt$$

$$\int \underline{[Dq]}\; O(\dot q)\; e^{\frac{i}{\hbar} S[\dot q]}$$

manifestly invariant under
$\underline{q(t)} \to q(t) + \underline{q_0}$
const

Let: $\bar q(t)$ be such that $\bar q(t=0) = 1$

$\bar q(t)$ is a representative of $[q]$



$$\int [Dq] = \underbrace{\int dq_0}_{=\infty} \int [D[q]]$$

in a non-abelian g.t.

$q_0 \hat{=} \quad U$ s.t. $U \neq U(x)$

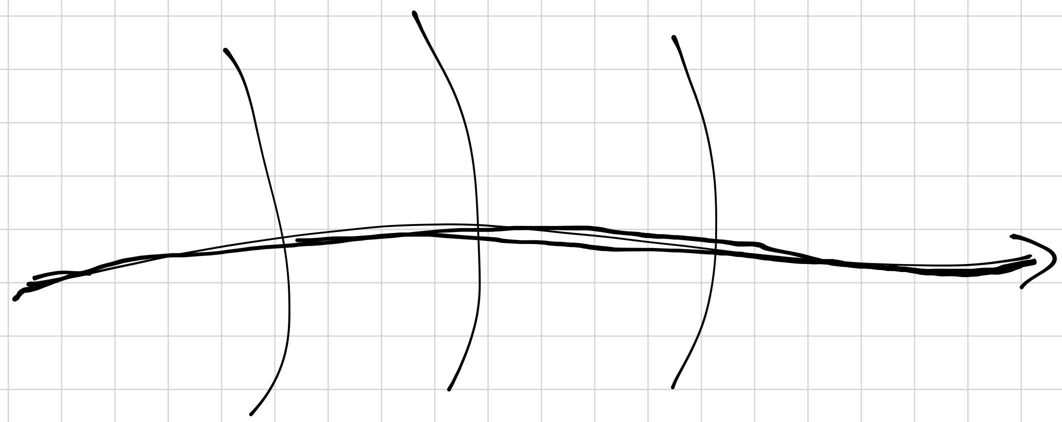
$q(t) + q_0 \hat{=} \quad A_\mu^U(x) = U A_\mu(x) U^{-1}$

In a gaug theory, however, $U = U(x)$

$\int [DU] = \mathrm{Vol}_G$ at each p.t.

Vol. gauge group $\cdot$ $|\mathrm{Vol}(\mathbb{R}^4)|$

Want to restrict the path integral to a single representative of each equivalence class.

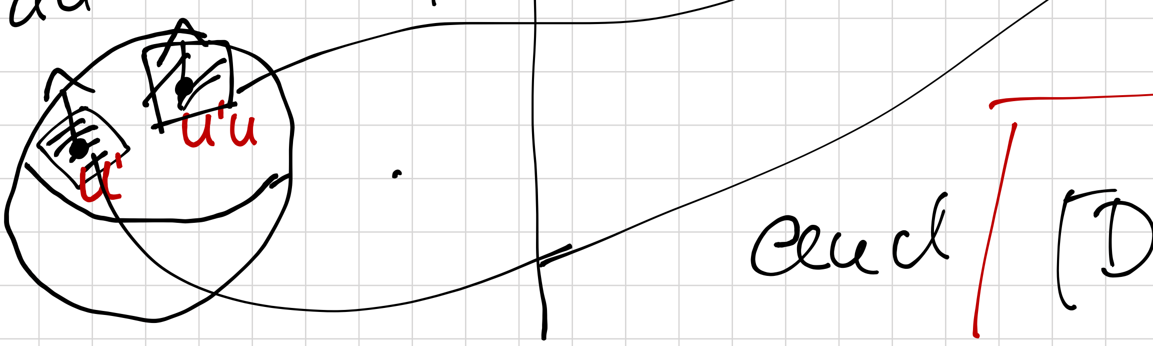


① choose a right-invariant measure : $U' \in G$, $U \in G$

Polcarski Weinberg Vol II

$$d(U'U) = dU'$$

dU



and

$$\boxed{[DU] \stackrel{\text{form.}}{=} \prod_{x \in \mathbb{R}^4} dU(x)}$$

is the corresponding functional measure

② Insert 1 into the path integral

$$1 = \Delta[A_\mu] \int [D\alpha]\, \delta[F(A_\mu^{\alpha})]$$

[cf $1 = f'(f^{-1}(0)) \int dq\, \delta(f(q))$]

$$\delta[F(A_\mu^{\alpha})] = \prod_{x \in \mathbb{R}^4} \delta(F(A_\mu^{\alpha})(x))$$

Example: $F(A_\mu)(x) = \partial_\mu A^\mu(x)$

Rep: $\partial_\mu A^\mu(x) = 0 \quad \forall x \Rightarrow$ Lorentz gauge.