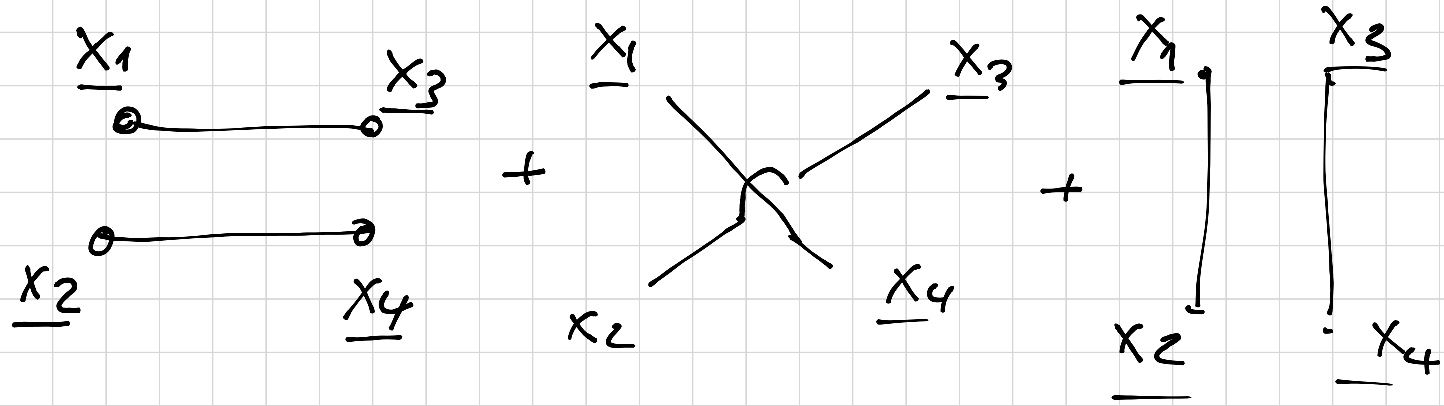
Normal ordening : consider

$$\langle \phi(\underline{x}_1)\phi(\underline{x}_2)\phi(\underline{x}_3)\phi(\underline{x}_4)\rangle = \underline{\underline{O(\lambda^0)}} + \cancel{O(\lambda)} \dots$$

$=$ 

$$= G_{\underline{x}_1 \underline{x}_3} \cdot G_{\underline{x}_2 \underline{x}_4} + G_{\underline{x}_1 \underline{x}_4} G_{\underline{x}_2 \underline{x}_3} + G_{\underline{x}_1 \underline{x}_2} G_{\underline{x}_3 \underline{x}_4}$$

$O(\lambda)$:

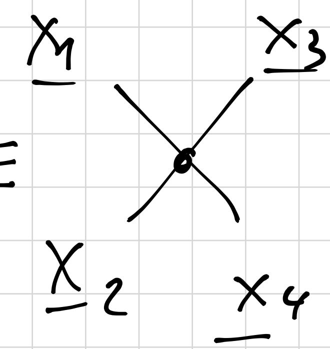
$$\langle \phi(\underline{x}_1)\cdots\phi(\underline{x}_4)\rangle = \frac{\lambda}{4!}\int d^4y\, \langle \phi(\underline{x}_1)\cdots\phi(\underline{x}_4):\phi^4(y):\rangle$$

$G_{\underline{y}\underline{y}}$

$$\left[\int d^4y\, \phi(\underline{x}_1)\phi(\underline{x}_2)\phi(\underline{x}_3)\phi(\underline{x}_4):\phi(\underline{y})\,\phi(\underline{y})\,\phi(\underline{y})\,\phi(\underline{y}):\right]$$

$G_{\underline{x_3}\underline{y}}$

$$= \lambda \int \underline{G_{\underline{x}_1\underline{y}}\, G_{\underline{x}_2 y}\, G_{\underline{x}_3\underline{y}}\, G_{\underline{x}_4\underline{y}}}\; d^4y \equiv$$



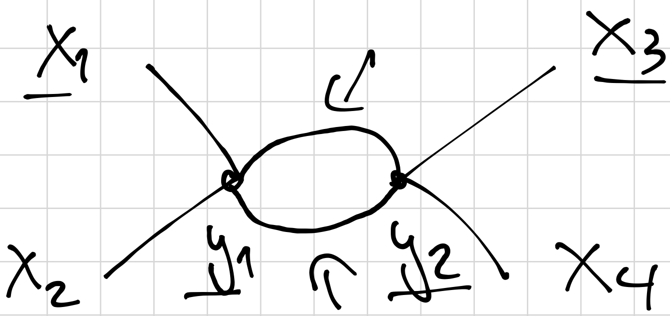
Renormalisation :

$\mathcal{O}(\lambda^2)$: $\quad \langle \quad \rangle = \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \int d^4y_1\, d^4y_2 \;\cdot$

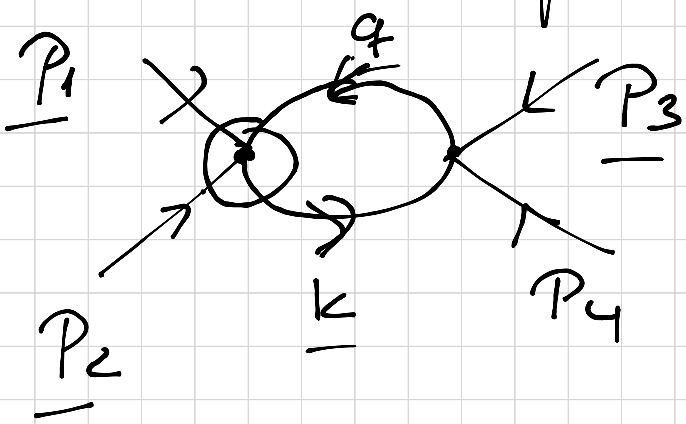
$$\langle \phi(\underline{x}_1)\cdots\phi(\underline{x}_4) :\phi(\underline{y}_1)\,\phi(\underline{y}_1)\,\phi(\underline{y}_1)\,\phi(\underline{y}_1)::\phi(\underline{y}_2)\,\phi(\underline{y}_2)\,\phi(\underline{y}_2)\,\phi(\underline{y}_2): \rangle$$

$$= \frac{1}{2}\,\#\int d^4y_1\, d^4y_2\;\; G_{\underline{x}_1\underline{y}_1}\, G_{\underline{x}_2 y_2}\, \boxed{G_{\underline{y}_1 y_2}\, G_{\underline{y}_1\underline{y}_2}}\, G_{\underline{x}_3\underline{y}_2}\, G_{\underline{x}_4\underline{y}_2}$$

$\equiv$



In momentum space:



$$= \frac{1}{p_1^2+m^2}\,\frac{1}{p_2^2+m^2}\,\frac{1}{p_3^2+m^2}\,\frac{1}{p_4^2+m^2}\cdot$$

$$\lambda^2\int d^4k\,d^4q\;\delta^4(p_1+p_2+q-k)\;\delta^4(p_3+p_4+k-q)\cdot$$

$$\cdot\,\frac{1}{k^2+m^2}\,\frac{1}{q^2+m^2}$$

$$= \prod_{i=1}^{4}\frac{1}{p_i^2+m^2}\,\lambda^2\,\delta(p_1+p_2+p_3+p_4)\int\frac{1}{(k^2+m^2)}\,\frac{1}{((k-p_1-p_2)^2+m^2)}\,d^4k$$

$$\rightarrow\ \#\int_0^{\Lambda=\text{cut-off}}|k|^3\,d|k|\;\frac{1}{|k|^2\left(1+\frac{m^2}{k^2}\right)}\;\frac{1}{|k|^2\left(1-\frac{2(p_1+p_2)k}{|k|^2}+\frac{(p_1+p_2)^2}{|k|^2}+\frac{m^2}{|k|^2}\right)}$$

$$\sim\ \#\int_0^{\Lambda}\frac{d|k|}{|k|}\left(1+O\left(\frac{1}{|k|}\right)\right)\ \sim\ \log(\Lambda)$$

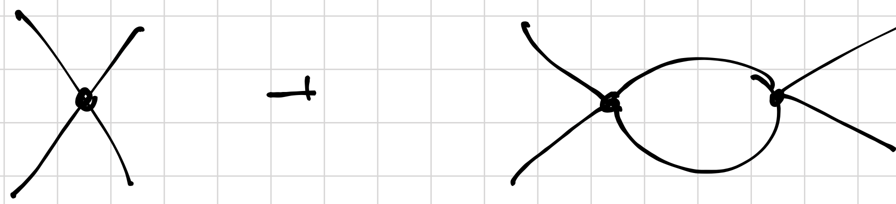
Lagrangian coupling

Cure: Define $\boxed{\lambda = \lambda_R - \# \lambda_R^2 \ln(\Lambda)}$

small

physical coupling

$O(\lambda)$ $\qquad\qquad\qquad$ $O(\lambda^2)$



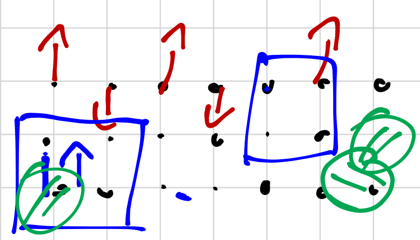
$\lambda_R \cdot \text{finite}$

$- \# \lambda_R^2 \log(\Lambda) \cdot \text{finite}$ $\qquad$ $\lambda_R^2 \log(\Lambda)$

cancel

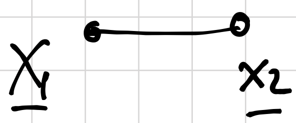
Renormalisation

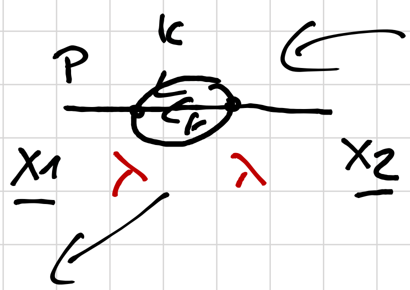
Interpretation in Stat mech



$\langle \phi(\underline{x}_1)\,\phi(\underline{x}_2)\rangle$

$$= G_{\underline{x}_1 \underline{x}_2} + \#\,\lambda^2 \int d^4y_1\, d^4y_2\; G_{\underline{x}_1 \underline{y}_1} \left(G_{\underline{y}_1 \underline{y}_2}\right)^3 G_{\underline{x}_2 \underline{y}_2}$$





2-loop

$\delta m \sim \lambda^2$

in momentum space:

$$\int d^4k\, d^4q \;\frac{1}{(k^2+m^2)(q^2+m^2)((\underline{k}+\underline{q}+p)^2+m^2)}$$

power counting: $8 - 6 = 2$

Path integral for fermions

dimension

$$S[\psi] = \int \bar{\psi}(i\not{\partial} - m)\psi \, d^D x$$

① "reduction" $D = 0$ $\quad \eta$: spinor (instead of ψ)

$$S[\eta] = -\bar{\eta} m \eta$$

η is an anticommuting variable

(behaves like differential forms: $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \eta \qquad \bar{\eta}$

$$\eta\bar{\eta} = -\bar{\eta}\eta \; ; \quad \eta^2 = \bar{\eta}^2 = 0$$

$\eta, \bar{\eta}$ are elements of a Grassmann algebra. Λ

$\uparrow$ anticommuting.

$\uparrow$ addition, multiplication

$$f: \Lambda \to \Lambda$$

$$\eta, \bar{\eta} \mapsto f_0 + f_1 \eta + f_2 \bar{\eta} + f_3 \eta \bar{\eta}$$

$f_0, \dots f_3 \in \mathbb{R} \, (\mathbb{C})$ even

Integration: linear functional on Λ:

<u>Def:</u> $\int d\eta \, \eta \equiv 1$; $\int d\eta \, 1 = 0$

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Def: $\int d\eta \, \eta \equiv 1 \; ; \; \int d\eta \, 1 = 0$

$$= \frac{\partial}{\partial \eta} \eta = 1 \quad \Big| \quad \frac{\partial}{\partial \eta} 1 = 0$$

n.b.: $\int d\eta \, \partial_\eta f(\eta) = 0$

$\quad \llcorner f = f_0 + f_1 \eta$

$\quad = f_1$

$\quad = 0$

no boundary

Rem:

① $\int f(\eta,\bar\eta) = \int d\eta\, d\bar\eta \underbrace{f(\eta,\bar\eta)}_{f_0 + f_1\eta + f_2\bar\eta + f_3\eta\bar\eta} = \partial_\eta \partial_{\bar\eta} f$

$$= -\partial_{\bar\eta}\partial_\eta f(\eta,\bar\eta) = -\int d\bar\eta\, d\eta\, f(\eta,\bar\eta)$$

② change of variables: $\eta' = a + b\eta'$

$a, b \in \mathbb{R},\ b \neq 0$

$$\int d\eta' = \int d\eta \left(\frac{\partial \eta'}{\partial \eta}\right)^{-1}$$

← $d\eta$ ~> differentiation w.r.t η

← Jacobian

Gaußian Integral

$$\eta_1, \dots, \eta_n, \bar\eta_1, \dots \bar\eta_n \in \Lambda$$

$$A = A^+ \in M_{n\times n} \quad \text{matrix}$$

$$\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\int d\eta_1 d\bar\eta_1 \cdots d\eta_n d\bar\eta_n \, \underbrace{e^{(\bar\eta, A\eta)}}_{\text{expand}} = \det(A)$$

(cf: $\frac{1}{\sqrt{\det(A)}}$ for bosonic gauß)

$$= \int d\eta_1 \cdots d\bar\eta_n \left(1 + \bar\eta A \eta + \frac{1}{2}(\bar\eta A \eta)^2 + \cdots \frac{1}{n!}(\bar\eta, A\eta)^n\right)$$

choose an eigenbasis of A ; $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$\int d\tilde\eta_1 d\bar{\tilde\eta}_1 \cdots d\tilde\eta_n d\bar{\tilde\eta}_n \left(\overbrace{\bar{\tilde\eta}_1 \lambda_1 \tilde\eta_1}^{\text{even}} \cdots \bar{\tilde\eta}_n \lambda_n \tilde\eta_n \right)$$

$$\lambda_1 \cdot \ldots \cdot \lambda_n = \det(A)$$

Return to the Dirac action:

$$\int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \quad ; \quad \text{Expand } \psi(x) = \sum_n \underline{\eta_n}\, \psi_n(x)$$

Grassmann

Eigenf.

even

cf. $f_0, f_1 \cdots f_s$

Euclidean formulation:

$$S_E(\psi, \psi^*) = \int d^4x\, \psi^*(\underline{x})\,(i\not{\partial} - m)\,\psi(\underline{x})$$

rep.: $\bar{\psi} = \psi^\dagger \gamma^0 \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\underbrace{g^{\mu\nu}}_{(1,-1,\cdots -1)}$

Minkowski

New: in Euclidean signature ψ, ψ^* are independent variables.

Expand: $\psi(\underline{x}) = \sum_n \eta_n \psi_n(\underline{x})$;

$$(i\not{\partial} - m)\psi_n(\underline{x}) = \lambda_n \psi_n(\underline{x})$$

$$(\psi_n^*, \psi_m) = \int d^4x\, \psi_n^*(\underline{x})\, \psi_m(\underline{x})$$

$$S_E(\psi^*, \psi) = \sum_n \bar{\eta}_n \eta_n \lambda_n$$

$$\int [D\psi, \psi^*]\, e^{\int \psi^* (i\not{\partial} - m)\psi + \int \psi^*(\underline{x})\alpha(\underline{x}) + \int \alpha^*(\underline{x})\psi(\underline{x})}$$

$$\alpha(x) = \sum_n \alpha_n f_n(x)$$

$$= \int \prod_n d\eta_n d\bar{\eta}_n \; e^{\sum_n \lambda_n \bar{\eta}_n \eta_n + \eta_n^* \alpha_n + \alpha_n^* \eta_n}$$

complete the square:

$$\psi^*(\underline{x}) \longrightarrow \tilde{\psi}^*(\underline{x}) = \psi^*(x) - [\alpha^* (i\not{\partial} - m)^{-1}](\underline{x})$$

$$\psi(\underline{x}) \longrightarrow \tilde{\psi}(\underline{x}) = \psi(\underline{x}) - [(i\not{\partial} - m)^{-1}\alpha](\underline{x})$$

$$= \prod_n \lambda_n \; e^{-\underbrace{\int \alpha^* (i\not{\partial} - m)^{-1} \alpha}_{\int d^4x\, d^4y\, \alpha^*(\underline{y})\, K(\underline{y}\, \underline{x})\, \alpha(\underline{x})}}$$

with $(i\not{\partial}_{\underline{x}} - m)\, K(\underline{x}, \underline{y}) = \underline{\underline{\mathbb{1}_{4\times4}}}\, \delta^4(\underline{x} - \underline{y})$