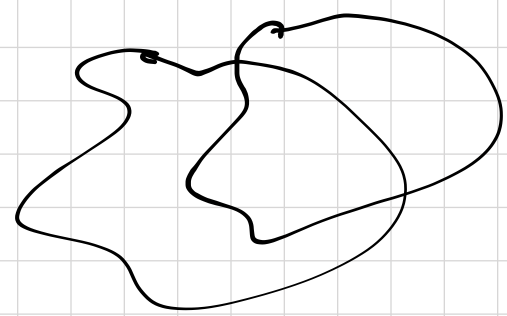
Rep: Monday:

① QM: $\mathrm{tr}\left(e^{-\beta H}\right) = \int [Dq]\, e^{-S_E[q(\tau)]}$

$$\boxed{t \in [0, T = -i\beta\hbar]}$$



② Field theory $q(\tau) \rightsquigarrow \phi(x)$ order parameter field

$$\boxed{Z[J] = \int [D\phi]\, e^{-S_E[\phi] + \int J\phi}}$$

$$S_E[\phi] = \int d^Dx \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right)$$

$\boxed{a(\beta) = m^2}$; $c(\beta) = \frac{1}{2}$, $b(\beta) = \frac{\lambda}{4!}$

Rem: 1 This is just classical statistical physics for $\phi(\underline{x})$ some ord. param. field.

Thm: (Osterwalder, Schrader)

classical stat mech $\equiv$ QFT
under certain assumptions which hold only for the gaußian field:

$$S_E[\phi] = \int \frac{1}{2}\left((\nabla\phi)^2 + m^2\phi^2\right)$$

2) We only know how to calculate Gaußian integrals

$\longrightarrow$ perturbation theory

$$S_E[\phi] = \underbrace{\int \frac{1}{2}\phi(-\nabla^2+m^2)\phi}_{S^{(2)}[\phi]} \underbrace{+\frac{\lambda}{4!}\phi^4}_{S_{int}[\phi]}$$

$\lambda \ll 1$

Compute $\langle \phi(\underline{x}_1)\, \phi(\underline{x}_2) \rangle =$

$$= \frac{1}{Z} \int [D\phi]\, e^{-S^{(2)}[\phi]}\, \phi(\underline{x}_1)\, \phi(\underline{x}_2) \cdot$$

$$\cdot \left(1 - \frac{\lambda}{4!} \int :\phi^4(x):\, d^D x \right.$$

$$+ \frac{1}{2} \frac{\lambda^2}{(4!)^2} \int d^D x\, d^D y :\phi^4(\underline{x})::\phi^4(y):$$

$$\left. + \dots \right)$$

$$Z = \int [D\phi]\, e^{-S^{(2)}[\phi]} \left(1 - \frac{\lambda}{4!} \int :\phi^4: d^D \dots \right)$$

$O(\lambda^0)$:

$$\langle \phi(\underline{x}_1)\,\phi(\underline{x}_2)\rangle = \frac{1}{Z_0[J]\big|_{J=0}} \frac{\delta}{\delta J(\underline{x}_2)} \frac{\delta}{\delta J(\underline{x}_1)} Z_0[J]\Big|_{J=}$$

$$Z_0[J] = \int [D\phi]\, e^{-S^{(2)}[\phi] + \int J\phi}$$

$$= \int [D\phi]\, e^{-\frac{1}{2}\int \phi(-\Delta + m^2)\phi + \int \underline{\underline{J\phi}}}$$

complete the square

$$Z_0[J] = \int [D\phi]\, e^{-\int \frac{1}{2}\Big(\overbrace{\phi(\underline{x}) - \int G_{\underline{x}\underline{y}_1} J(y_1)}^{\equiv \tilde{\phi}(\underline{x})}\Big)(-\Delta_{\underline{x}} + m^2)\cdot \left(\phi(\underline{x}) - \int G_{\underline{x}\underline{y}_2} J(\underline{y}_2)\right) + \boxed{\frac{1}{2}\iiint J(y_1)\, G_{\underline{y}_1\underline{x}} \underbrace{(-\Delta_x + m_2)\, G_{\underline{x}\underline{y}_2}}_{1} J(\underline{y}_2)}}$$

$$[D\phi] = [D\tilde{\phi}]$$

with $\underline{(-\Delta_{\underline{x}} + m^2)\, G_{\underline{x}\cdot\underline{y}}} = \delta^D(\underline{x} - \underline{y})$

solu: $\displaystyle G_{\underline{x}\underline{y}} = \sum_n \frac{\phi_n(\underline{x})\,\phi_n(\underline{y})}{\lambda_n}$

$$Z_0[J] = \underbrace{\int [D\tilde{\Phi}]\, e^{-\frac{1}{2}\int \tilde{\Phi}(-\Delta + \omega^2)\tilde{\Phi}}}_{Z_0[J=0]} \cdot e^{\frac{1}{2}\iint J(\underline{x})\, G_{\underline{x}\underline{y}}\, J(\underline{y})}$$

Thus:

$$\langle \phi(\underline{x}_1)\,\phi(\underline{x}_2)\rangle = \frac{1}{Z}\,\frac{\delta^2 Z}{\delta J(\underline{x}_1)\,\delta J(\underline{x}_2)}\bigg|_{J=0}$$

$$= \underbrace{G_{\underline{x}_1\underline{x}_2}}_{\langle \phi(\underline{x}_1)\phi(\underline{x}_2)\rangle|_{O(\lambda^0)}} + O(\lambda)$$

To see the connection of $G_{\underline{x_1}\,\underline{x_2}}$ with the propagator in QFT we first Fourier expand:

$$G_{\underline{x_1}\,\underline{x_2}} = \int \frac{d^D p}{(2\pi)^4} \, \frac{1}{\underline{p}^2 + m^2} \, e^{-i\,\underline{p}\cdot(\underline{x}-\underline{y})}$$

(Volume) $\to \infty$

$(-i)$

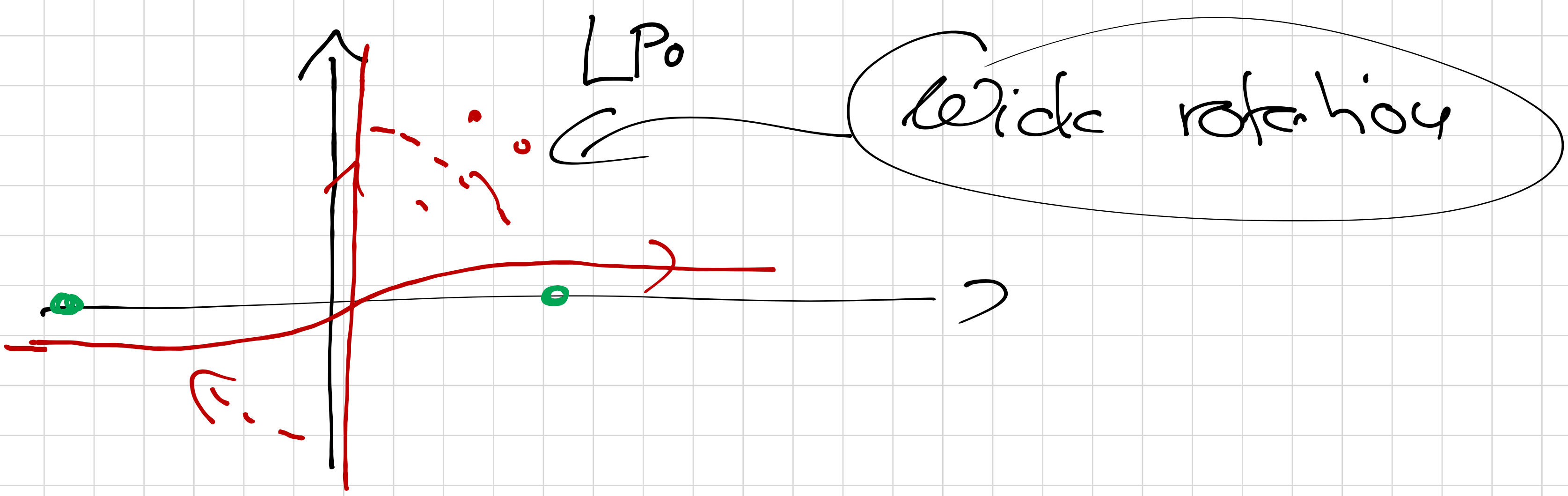
$-p_0^2 + \underline{p}^2 + m^2$

Goto Minkowski signature:

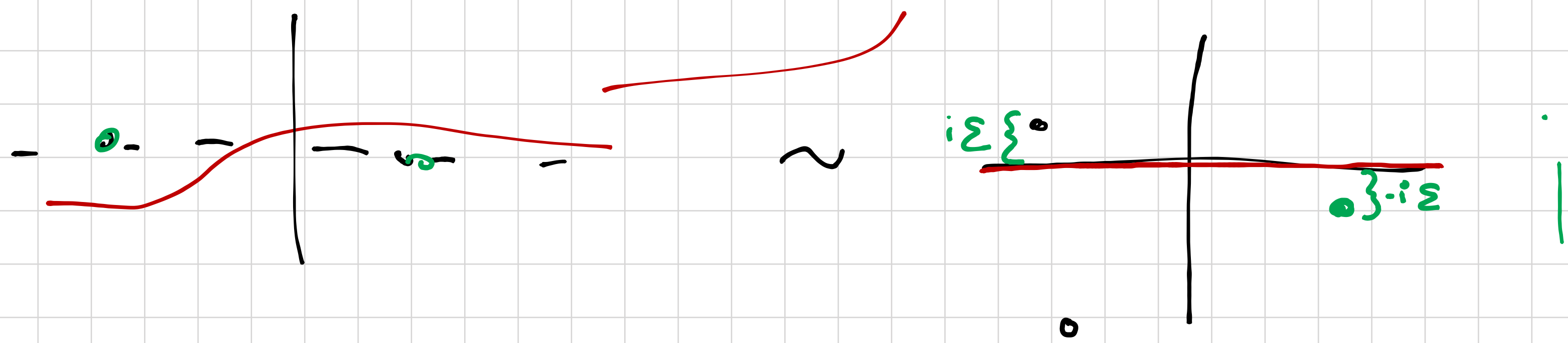
$$\underbrace{\tau}_{\text{Eucl. time}} = \underbrace{e^{\frac{\pi i}{2}}}_{i} \frac{t}{\hbar} \;; \quad p_\tau \sim \partial_\tau = e^{-\frac{\pi i}{2}} p_0$$

$$\int d^4p \, \frac{i e^{-ip(\underline{x}-\underline{y})}}{\underline{p}^2+m^2} \to -i \int d^3p \int dp_0 \, \frac{1}{-p_0^2+\underline{p}^2+m^2} \, e^{-i \overbrace{p\cdot(x-y)}^{p_0x^0-\underline{p}\,\underline{x}}}$$

p_0 integral has a pole $p_0 = \pm\sqrt{\underline{p}^2+m^2}$



$$G(\underline{x}, x_0, \underline{y}, y_0) = \frac{1}{(2\pi)^4} \int d^3\underline{p} \int_C dp_0 \, \frac{i}{p_0^2 - \underline{p}^2 - m^2} \, e^{-ip(x-y)}$$



$$= \frac{1}{(2\pi)^4} \int d^4p \, \frac{i}{p_0^2 - \underline{p}^2 - m^2 + i\varepsilon}$$

→ Feynman $i\varepsilon$ prescription

$$= \langle 0 | T \{ \hat\phi(\underline{x}, x_0), \hat\phi(\underline{y}, y_0) \} | 0 \rangle$$

Time ordering.

$$= \Theta[x_0 - y_0] \langle 0 | \hat\phi(\underline{x}, x_0) \, \hat\phi(\underline{y}\, y_0) | 0 \rangle$$
$$+ \, \Theta(x_0 - x_0) \langle 0 | \hat\phi(\cancel{\underline{x}}\, y_0) \, \hat\phi(\underline{x}, x_0) | 0 \rangle$$

Interpretation at the level of the path integral:

$$S_E[\phi] = \int \underbrace{d\tau}_{i\,dt}\, d^3\underline{x}\, \Big(\underbrace{(\partial_\tau \phi)^2}_{-(\partial_t\phi)^2} + (\nabla_{\underline{x}}\phi)^2 + m^2\phi^2 \Big)$$

$$\overset{\text{Wick}}{\longrightarrow} -i\int dt\, d^3x\, \Big((\partial_t\phi)^2 - (\nabla_{\underline{x}}\phi)^2 - m^2\phi^2 \Big)$$

momentum space

$$\longrightarrow -i \int d^4p\; \tilde{\phi}(-p)\big(p_0^2 - \underline{p}^2 - m^2 + i\varepsilon\big)\phi(p)$$

$$\int [D\phi]\, e^{-S_E[\phi]} \overset{\text{Wick}}{\longrightarrow} \int [D\phi]\, e^{\frac{i}{\hbar}S[\phi] - \varepsilon\int\phi^2}$$

fall-off at large values of ϕ.

Interactions

$$S[\phi] = S^{(2)}[\phi] \underbrace{- \frac{\lambda}{4!}\int d^4x :\phi(x)^4:}_{= S_{int}}$$

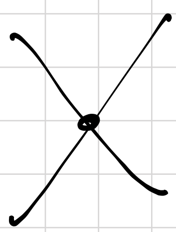
$$Z[J] = \int [D\phi]\, e^{\frac{i}{\hbar} S^{(2)}[\phi]} \Big(1 + \underbrace{\frac{i}{\hbar} S_{int}(\phi)}_{-\frac{i\lambda}{\hbar\, 4!}\int d^4x\, \underbrace{\phi^4}_{\int d^4p_1 \cdots d^4p_4\, \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_4)\, \delta(\sum_i p_i)}} - \frac{1}{2\hbar^2} S_{int}^2 + \cdots\Big)$$

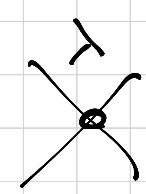
$$-\frac{i\lambda}{\hbar\, 4!}\,\delta^4\Big(\sum_{i=1}^{4} p_i\Big)\, \underbrace{4!}_{\text{symmetry}}$$

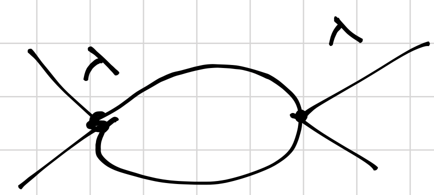
vertex interaction

Feynman rules:

 $= \dfrac{i\hbar}{p^2 - m^2 + i\varepsilon}$ propagator

 $= \dfrac{i\lambda}{\hbar}\,\delta^4\left(\sum_i p_i\right)$ vertex

$\langle \phi(p_1) \cdots \phi(p_4) \rangle =$  $+$ 

$+$  $+ \ldots$