Rep: QED:

$$I_{QED}(\psi, \bar{\psi}, A_\mu) =$$

$$\int d^4x \left\{ i\bar{\psi}(\not{D} + im)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right\}$$

$$\not{D} = \gamma^\mu D_\mu \, ; \quad D_\mu = \partial_\mu + igA_\mu$$

$$\rightarrow A_\mu j^\mu = L_{int}$$

Rep: non-abelian g.T.

A_μ replaced by $A_\mu{}^a T_a$

generators of $\mathfrak{g}$

$\mathfrak{g}$: Lie algebra.

$$[T_a, T_b] = i \underbrace{C_{ab}{}^c}_{\text{structure const.}} T_c$$

structure const. $\rightsquigarrow$ define $\mathfrak{g}$

$C_{ab}{}^c = 0 \rightsquigarrow$ QED abelian

Rep:

$$ig A_\mu^{(u)} = U^{+-1}_{(x)} \partial_\mu U(x) + i U^{+-1} A_\mu U^{-1}$$

$$\simeq i\partial_\mu \theta^a T_a + i[A_\mu, \theta^a T_a]$$

$$U(x) \simeq 1 - i\theta^a(x) T_a$$

L_{int} : $A_\mu^a \, j_a = A_\mu^a \underbrace{\bar{\Psi} \gamma^\mu T_a \Psi}_{= j_a}$

$L_{matter} = L_m$ ✓

$\cup$

L_{int} ✓

L_{gauge} : $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$

not gauge inv ↯

$$G_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$$

$$G^{(u)}_{\mu\nu} = \frac{1}{ig}\left[D^{(u)}_\mu, D^{(u)}_\nu\right]$$

$$= \frac{1}{ig}\left[U^{+-1} D_\mu \underbrace{\cancel{U^{-1}} \cancel{U^{+-1}}}_{\mathbb{1}} D_\nu U^{-1}\right]$$

$U^+ = U^{-1}$ (unitary group)

$$\boxed{G^{(u)}_{\mu\nu} = \frac{1}{ig} U [D_\mu, D_\nu] U^{-1} = U G_{\mu\nu} U^{-1}}$$

$$G_{\mu\nu} = \frac{1}{ig}[\underbrace{D_\mu, D_\nu}] \equiv G^a_{\mu\nu}\, \underline{\underline{T_a}}$$

~~cf Exer: sheet $\underline{\underline{2}}$~~

$$[T_a, T_b] = i C_{ab}{}^c T_c$$

$$L_g = -\frac{1}{4} \underset{?}{\underline{\underline{B_{ab}}}} G^{a}_{\mu\nu} G^{\mu\nu b} = \cancel{G^a_{\mu\nu} G^{\mu\nu b} \underbrace{T^a T^b}_{?}}$$

Solution: $\underbrace{B_{ab}}_{\text{metric}} \equiv - C_{ad}{}^c C_{bc}{}^d$ struct. const.

$$\underline{\underline{\angle}}\ \mathrm{tr}(T^{adj}_a T^{adj}_b)$$: Killing form

B_{ab} needs to be non-degenerate and pos. def.

cf. ED: Energy $= \oplus \int d^3x \, (\underline{E}^2 + \underline{B}^2)$

non abelian $\sim (\underline{\underline{B_{ab}}}\, \underline{E}^a \cdot \underline{E}^b + \cdots)$

B_{ab} non-degenerate $\Leftrightarrow$ $\mathfrak{g}$ is <u>semi-simple</u> = direct sum of <u>simple</u> Lie algebras. ($\rightsquigarrow$ Ilka Brunner's lectu.)

Book: Pokorski's book for chapter 0) motivation... and 1)

Summary:

quarks in fund. rep.

$A_\mu \in \mathfrak{g}$

$$\boxed{L(\psi, \bar\psi, A_\mu) = \bar\psi (i \not{D} - m)\psi - \frac{1}{4} B_{ab} G^a_{\mu\nu} G^{\mu\nu b}}$$

Examples: · QED Gauge group $G = U(1)$

QCD: " " $G = SU(3)$

Standard model Gaug. group $\underset{\text{colour}}{SU(3)} \times \underset{\text{flavour}}{SU(2)} \times U(1)$

more precisely: $SU(3) \times S(U(2) \times U(1))$

Rem: · mathematically there need to assume that the fermions are in the fundamental rep.

· For instance in SUSY theories all members of the SUSY multiplet must be in the same repr. $\rightarrow$ gluon boson
gluino fermion

2) Path integral quantisation

Sachs / Sen / Sexton "statistical mech"

Rep: QM:

$$\langle f | e^{-\frac{i}{\hbar} H T} | i \rangle = N \int_{q_i}^{q_f} \underbrace{[Dq]}_{T} \, e^{\frac{i}{\hbar} S[q(t)]}$$

$$e^{-\beta H} \qquad\qquad e^{-S_E[q(T)]}$$

$$\int \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$$

take $T \equiv -i \hbar \beta$

$$\langle f | e^{-\beta H} | i \rangle = \int_{q_i}^{q_f} [Dq]\, e^{-S_E[q(\tau)]}$$

imaginary time

$$S_E[q(\tau)] = \int_0^{\hbar\beta} \left(\frac{m}{2} q'(\tau)^2 + V(q) \right)$$

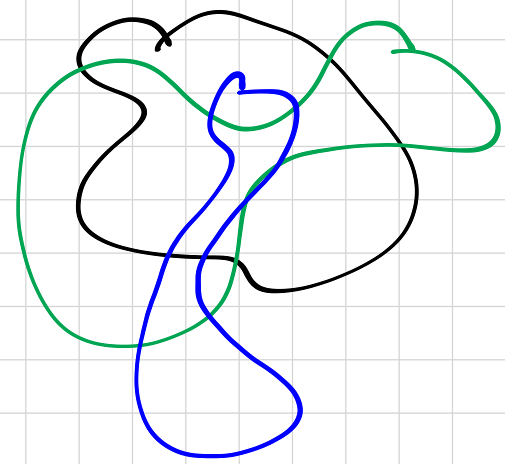
$$q'(\tau) = \frac{\partial}{\partial \tau} q(\tau)$$

$$Z[\beta] = \mathrm{tr}\left[e^{-\beta H}\right] = \int_{q:}^{q:} [Dq]\, e^{-S_E[q]}$$

$$= \sum_n \langle n | e^{-\beta H} | n \rangle$$

partition function in stat. mech.

trace in x-space.

$=$ 

$$Z[\beta] = e^{-\beta F}$$

F: free energy.

Field theory generalisation

follow the idea of the Landau free energy in S.M. $\sim$ Socks/Sen/Sexton

$\phi(\underline{x})$: order parameter field (e.g. magnetisation)

Effective "potential" for $\phi(\underline{x})$

source

$$F_L(\beta, \phi, \overset{\text{source}}{J}) = \int d^D_x \Big(a(\beta)\, \phi(\underline{x})^2 + b(\beta)\, \phi(\underline{x})^4 + \dots + c(\beta)\, \big(\underbrace{\underline{\nabla}\, \underline{\phi}}_{\text{fluctuation}}\big)^2 - J(\underline{x})\, \phi(\underline{x}) \Big)$$

fluctuation

near T_c (critical temp.) ϕ is small

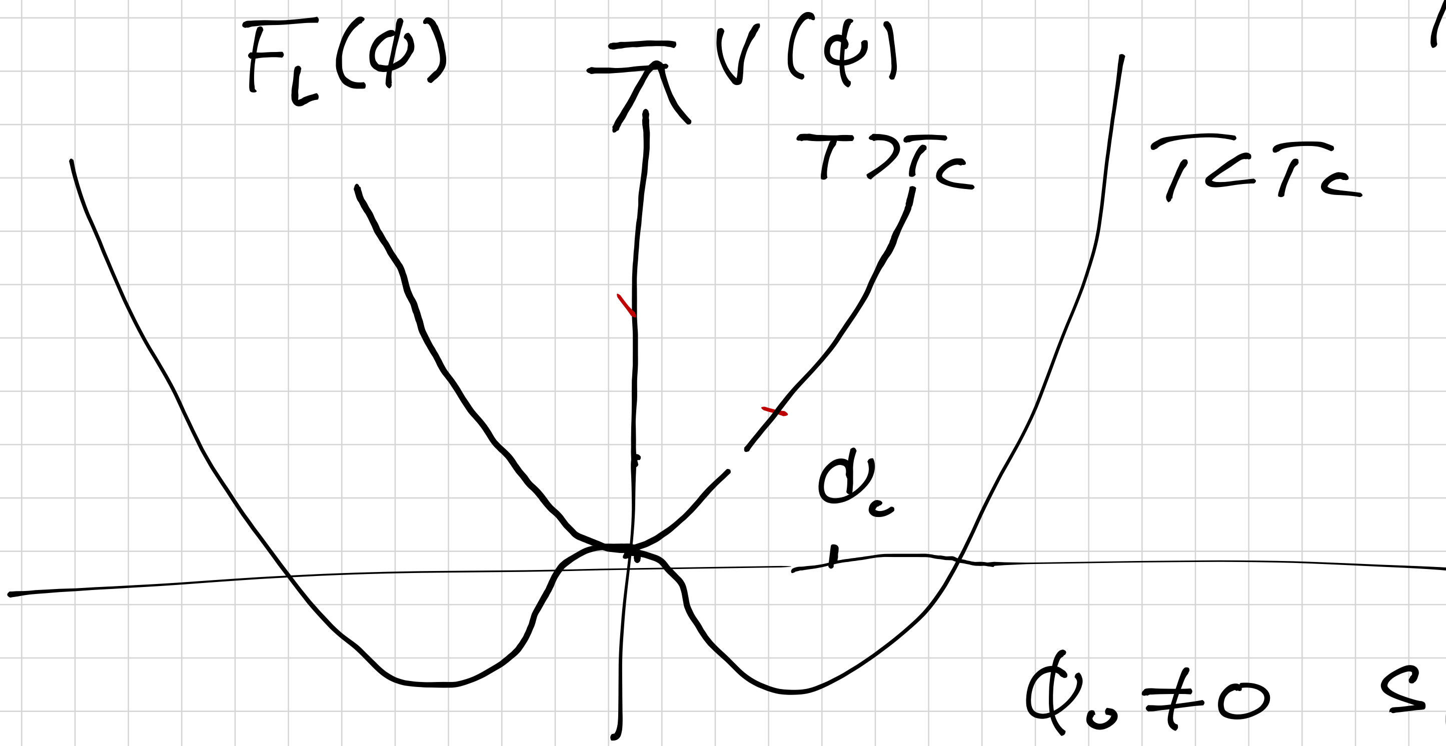
$a(\beta)$, $b(\beta)$, $c(\beta)$ are phenom. functions of (β)

near $T \simeq T_c$ and in mean field approx

$\phi(\underline{x}) = \phi_0$

$$a(\beta) = a_0\left(\frac{1}{\beta} - \frac{1}{\beta_c}\right)$$

$$= a_0(T - T_c)$$



ϕ

$\phi_0 \neq 0$ spont. magnetisation

Free energy (Landau):

$$F(\beta) = F_L(\beta, \phi, J)\Big|_{\substack{\phi=\phi_0 \\ J=0}}$$

Include fluctuations:

$$Z[J,\beta] = \underbrace{\int [D\phi]}_{\text{sum over fluctuations}} e^{-F_L(\beta,\phi,J)}$$

$$\phi(\underline{x}) = \phi_0 + \delta\phi(x)$$

Def'n of $[D\phi]$: assume finite volume:

$$\phi(x) = \sum_n \underline{\underline{c_n}} \phi_n(x) \; : \; (\underline{\nabla}^2)\phi_n = \lambda_n \phi_n$$

Inner product: $(\phi_n, \phi_m) = \int d^D x \, \phi_n(\underline{x}) \phi_m(\underline{x})$

$= \delta_{nm}$

then:

$$\int [D\phi] e^{-F_L(\beta, \phi,])} = \prod_n \int dc_n e^{-F_L(\phi, \beta,])}$$