If [couplings] ≥ 0 for all couplings then the [couplings] cannot increase the degree of divergence D and thus the only divergent diagrams that can diverge are those which contribute to $\Gamma^{(n)}$ s.t. $[\Gamma^{(n)}] > 0$. But all such divergences can be absorbed in a redefinition of the $\lambda_{n,p}$ provided we include all allowed couplings of pos dimensionand provided that all divergencies give rise to local counterterms. The latter property follows form the observation that differentiatiating sufficiently many times w.r.t. to the external momenta any loop diagram will be finite. Therefore the counterterms will be polynomial functions of the momenta. Thus a field theory with coupling constants of positive dimensions will be perturbatively renormaliseable. Conversely, a field theory with couplings of negative mass dimension will not be perturbatively renormaliseable.

Example: $S = \int \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - g_3\phi^3 - g_4\phi^4\right) d^4x$ $\qquad g_{\mu\nu} = \delta_{\mu\nu} + \varkappa h_{\mu\nu}$

Counterexamples $\int\left(\frac{1}{2}(\partial\phi)^2 - g_6\phi^6\right)$; $\quad \frac{1}{\varkappa^2}\int\sqrt{g}R = \int \underline{\nabla} h_{\mu\nu}\cdot\underline{\nabla} h_{\mu\nu} + \dots + \varkappa^n (h_{\mu\nu})^{n+2}$

Finally, let us show that Γ is indeed the generating functional of proper Feynman diagrams for a generic quantum field theory. Here we will follow Weinberg (Vol II). For this we consider the functional

$$\Gamma = W(J) - \Phi J$$

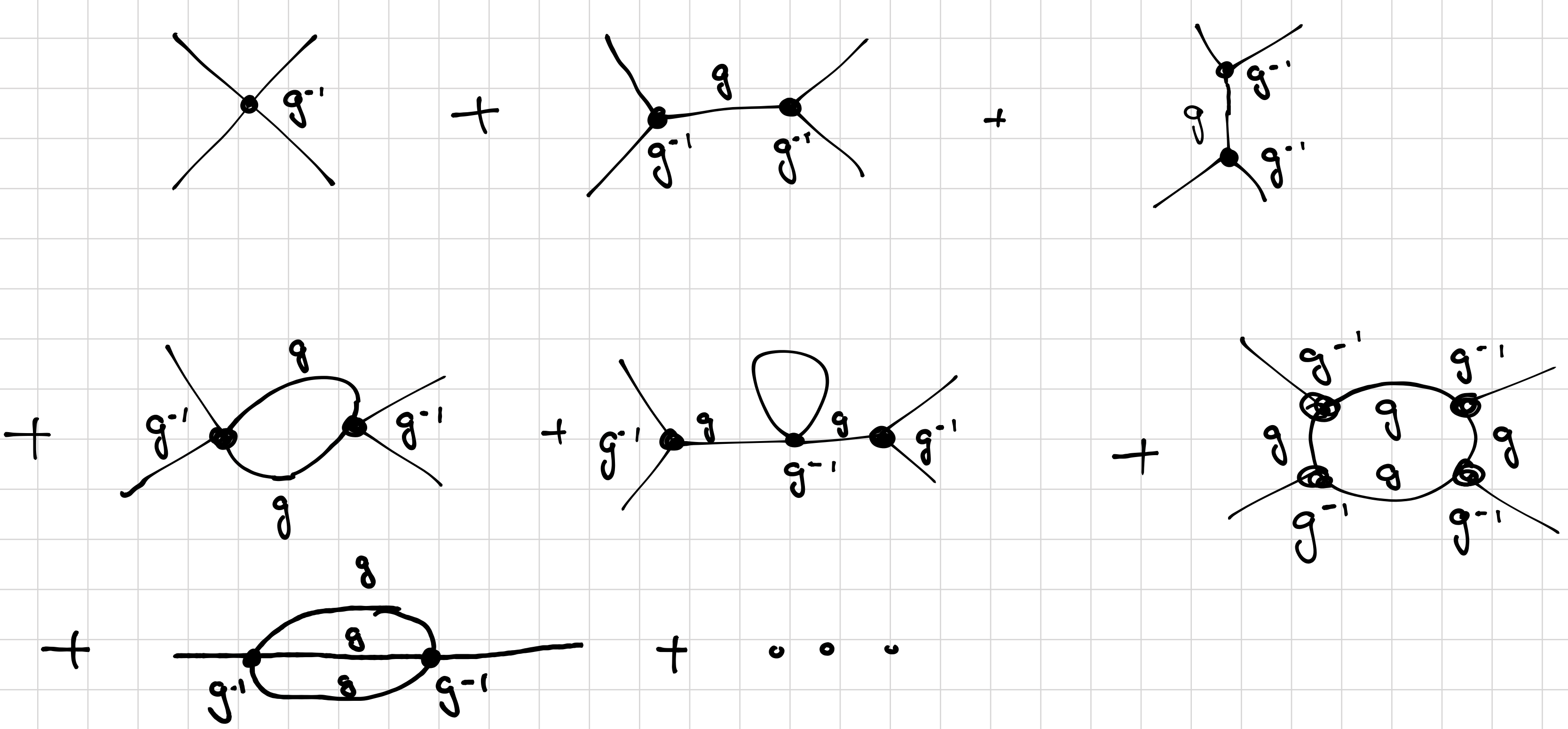
$$e^{iW_\Gamma(J,g)} := \int D[\Phi]\, e^{\frac{i}{g}\left\{\Gamma[\Phi] + \int \Phi J\right\}} \qquad (*)$$

("quantise the eff. action")

where g is some "coupling" constant. A graph with I internal lines and V vertices comes with a factor

$$g^{I-V}$$

with (Feynman) graphical representation: $\qquad$ e.g. $\Gamma = \frac{1}{g}\int d^4x\left(\Gamma^{(2)}\Phi^2 + \Gamma^{(3)}\Phi^3 + \dots\right)$ "inv. propag."



The number of loops, L is $L = I - V + 1$

Thus, the L-loop contribution to W_Γ is proportional for g^{L-1}. all together,

$$W_\Gamma(J, g) = \sum_{L=0}^{\infty} g^{L-1} W_\Gamma^{(L)}[J].$$

For the tree-level contribution to $W_\Gamma^{(0)}(J)$ we then take the limit of $(*)$ when $g \to 0$ where $(*)$ is dominated by the saddle point contribution,

$$e^{\frac{i}{g} W^{(0)}[J]} \propto e^{\frac{i}{g}\left\{\Gamma[\tilde{\Phi}] + \int \tilde{\Phi} J\right\}} \quad (\Delta)$$

where $\tilde{\Phi}$ is determined by

$$\frac{\delta \Gamma(\tilde{\Phi})}{\delta \tilde{\Phi}} = -J$$

Thus, the exponent on the r.h.s is just $W[J]$ (inverse Legendre transform). We then conclude the $W[J]$ is given by the <u>tree-level</u> diagrams obtained from $\Gamma(\Phi)$ and therefore $\Gamma(\Phi)$ must compute the proper diagrams.

B-V equation for Γ: In the absence of boundary terms in $\int D[\phi](\cdots)$ we have:

$$0 = \int [D\phi\phi^*]\, \Delta\left(f\, e^{\frac{i}{\hbar}S + i\int \phi J}\right)\Big|_{\phi^* = \frac{\delta\Psi}{\delta\phi}}$$

$$= \Big\langle \underbrace{(\Delta f) + \frac{i}{\hbar}(S,f)}_{=0 \text{ (f.g.i)}} - \frac{1}{2\hbar^2}\underbrace{\big((S,S) - 2i\hbar\Delta S\big)}_{=0 \text{ (ass)}} f$$

$$\underbrace{+ i(f, \phi J)}_{\substack{=0\\ \text{(ass.)}}} \left(f \neq f(\phi^*)\right) - \frac{1}{\hbar}(S, \phi J) \Big\rangle_J$$

$$\overset{f=1}{=} \Big\langle \frac{-1}{\hbar}\, s(\phi)\, J \Big\rangle_J = J \frac{(-1)}{\hbar} \underbrace{\langle s(\phi)\rangle_J}_{=: \frac{\delta_R \Gamma(\Phi, \chi^*)}{\delta \chi^*}}$$

$J = \frac{\delta_L \Gamma}{\delta \Phi}$ (by def'n of $\Gamma(\Phi)$)

(def'n of ext. of $\Gamma[\Phi]$ to $\Gamma[\Phi, \chi^*]$)

$$\overset{J = J(\phi, \chi^*)}{=} \frac{-1}{\hbar}\frac{\delta_R \Gamma}{\delta \chi^*}\frac{\delta_L \Gamma}{\delta \Phi} = \frac{1}{2\hbar}(\Gamma, \Gamma) \qquad (*)$$

$J = J(\phi, \chi^*) \rightsquigarrow \langle \phi \rangle_{J(\phi,\chi^*)} \overset{!}{=} \Phi$