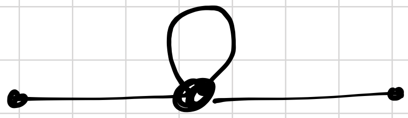
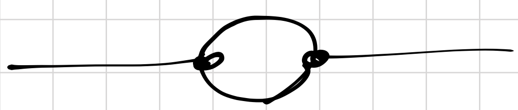
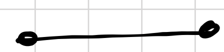
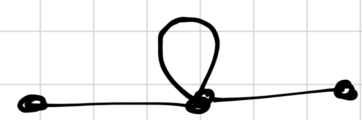
Next we compute the one loop contribution to $\Gamma^{(2)}$

For a $\lambda\phi^4$-theory the only one-loop diagram with 2 external legs is



For a $\lambda\phi^3$ we would also have

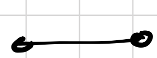
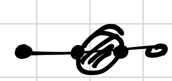
 at 1-loop

Thus $\left(\Gamma^{(2)}\right)^{-1}(p)$ =  + 

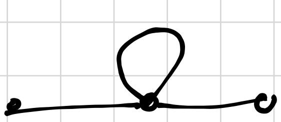
$\frac{i}{p^2-m^2}$

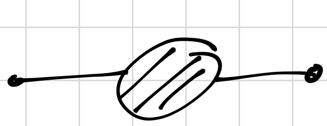
More generally such diagrams can be resummed as

$$\left(\Gamma^{(2)}(p)\right)^{-1} = \frac{i}{p^2-m^2-\text{}}$$ 

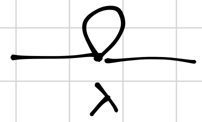
=  +  +  +

 + ...

where  is the 1-loop contribution to



Concretely,

with  $= -\frac{1}{2}\lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}$ $\quad(*)$

which is a divergent integral. In order to obtain a well defined result for $\Gamma^{(2)}$ we then reparametrise the <u>bare action</u> $S[\varphi]$ as

$$S[\varphi, m, \lambda] = \int \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_R^2\varphi^2 - \frac{\lambda_R}{4!}\varphi^4 \, d^4x$$

$$+\hbar\int \frac{1}{2}\delta z(\partial\varphi)^2 - \frac{1}{2}\delta m^2\varphi^2 - \frac{\delta\lambda}{4!}\varphi^4 \, d^4x$$

where m_R, λ_R and φ are the renormalised "physical" variables and $m^2 = m_R^2 + \delta m^2$ and $\lambda = \lambda_R + \delta\lambda$ are formal parameters that are determined order by order in $\hbar$ in such a way that the physical observables are well defined. In other words, $S[\varphi, m, \lambda]$ is but a formal object that has to be defined in each order in perturbation theory. In order to determine δm^2 we then first regularise. For instance in <u>dimensional regularisation</u>

$$(*) = -\frac{1}{2}\lambda_R\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m_R^2} = -\frac{1}{2}\lambda_R\frac{\mu^{4-d}}{(2\pi)^d}\,\mathrm{Vol}(S^{d-1})\int_0^\infty \frac{dk\, k^{d-1}}{k^2-m_R^2}$$

where we used that, in d-dimensions λ has dimension 4-d and we took λ_R to be dimensionless. Then with $\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$

$$(*) = \frac{m_R^2 \lambda_R}{2(4\pi)^{d/2}} \left(\frac{\mu}{m_R}\right)^{4-d} \Gamma\left(1 - \frac{d}{2}\right)$$

For $\varepsilon = 4-d \ll 1$ this gives

$$(*) \sim -\frac{m_R^2 \lambda_R}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right] + O(\varepsilon)$$

where $\gamma = \lim_{z \to 0}\left(\Gamma(z) - \frac{1}{z}\right)$. In the minimal subtraction (or MS) scheme we then set

$$\boxed{\delta m^2 = -\frac{m_R^2 \lambda_R}{16\pi^2}\frac{1}{\varepsilon}}$$

at first order in $\hbar$.

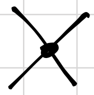
We note in passing that the dimensionfull parameter μ was introduced merely to have the correct canonical dimension so, physical quantities should not depend on the value of numerical value of μ. Independence of $(*)$ then implies that

$$-\frac{m_R^2 \lambda_R}{32\pi^2}\left[-\gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right] \text{ is independent}$$

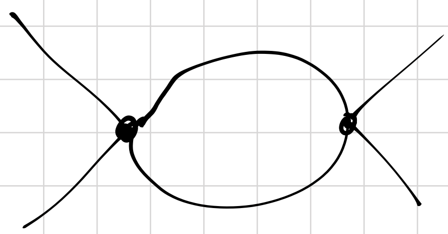
of μ which gives rise to the renormalisation group equations

For a ϕ^4-theory $T^{(3)}$ vanishes since all 3-point functions vanish in this theory (exercise)

The classical contribution to $\Gamma^{(4)}$ is just

$S^{(4)}$ =  $= \frac{\lambda}{4!}$

We can similarly compute the 1-loop correction to $\Gamma^{(4)}$ by computing the 4-pt function:



at zero external momenta

given by

$$\frac{\lambda_R^2 \mu^{4-d}}{2} \int \frac{d^dk}{(2\pi)^k} \frac{1}{(k^2 - m_R^2)^2} + t + u\text{-channel}$$

dim reg

$$= \frac{3}{2} \lambda_R^2 \left(\frac{\mu}{m_R}\right)^{4-d} \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(2)}$$

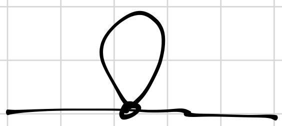
$$= \frac{3\lambda_R^2}{(4\pi)^{d/2}} \left(\frac{\mu}{m_R}\right)^{4-d} \Gamma\left(2-\frac{d}{2}\right) \simeq \frac{\lambda_R^2}{32\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \lg\left(\frac{4\pi\mu^2}{m_R^2}\right)\right) + O(\varepsilon)$$

Thus, in MS:

$$\boxed{\delta\lambda = \frac{\lambda_R^2}{32\pi^2}\frac{2}{\varepsilon}}$$

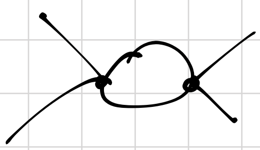
Renormalisable Field Theory:

When is a field theory renormaliseable? For this we have a look at the degree of divergence D of an individual loop diagram. For instance, neglecting all external momenta and masses,

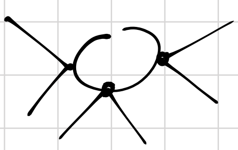


$$\sim \int^{\Lambda} \frac{d^4k}{k^2} \propto \Lambda^2 \quad : \quad D = 2$$

Λ: UV cut-off



$$\sim \int \frac{d^4k}{k^4} \propto \ln(\Lambda) \qquad D = 0$$



$$\sim \int \frac{d^4k}{k^6} \propto \Lambda^{-2} \qquad D = -2$$

More generally, dimensional analysis implies that for any diagram, contributing to the 1 PI vertex $\Gamma^{(n)}$

$$D = [\Gamma^{(n)}] - [\text{couplings}]$$

For instance, for a scalar field with $[\varphi] = \frac{d-2}{2}$ and

$$\mathcal{L}_{int} = \sum_{n \geq 3} \sum_{p} \lambda_{np}\, \partial^p \varphi^n$$

$$[\lambda_{np}] = d - n\frac{d-2}{2} - p$$

with $[\Gamma^{(n)}] = [\lambda_{n0}]$ (from tree level contribution to $\Gamma^{(n)}$)

if $\lambda_{n0} \neq 0$